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AN INTEGRATED FRAMEWORK FOR PROBABILISTIC OPTIMIZATION USING INVERSE RELIABILITY STRATEGY

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ABSTRACT

In this work, we propose an integrated framework for probabilistic optimization that can bring both the design objective robustness and the probabilistic constraints into account. The fundamental development of this work is the employment of an inverse reliability strategy that uses percentile performance for assessing both the objective robustness and probabilistic constraints. The percentile formulation for objective robustness provides an accurate probabilistic measure for robustness and more reasonable compound noise combinations. For the probabilistic constraints, compared to a traditional probabilistic model, the proposed formulation is more efficient since it only evaluates the constraint functions at the required reliability levels. The other major development of this work is a new search algorithm for the Most Probable Point of Inverse Reliability (MPPIR) that can be used to efficiently evaluate the performance robustness and percentile performance in the proposed formulation. Multiple techniques are employed in the MPPIR search, including the steepest decent direction and an arc search. The algorithm is applicable to general non-concave and non-convex functions of system performance with random variables following any continuous distributions. The effectiveness of the MPPIR search algorithm is verified using example problems. Overall, an engineering example on integrated robust and reliability design of a vehicle combustion engine piston is used to illustrate the benefits of the proposed method.

1. INTRODUCTION

Recent years have seen many developments for design under uncertainty. It is generally believed that without considering the impact of uncertainties, a design solution may either be sensitive to variations of system input which will lead to loss of system performance or lead to risk in violating critical design constraints. To accommodate uncertainties in engineering systems, methods of design under uncertainty have been applied increasingly. Among these are Robust Design [1-2] and Reliability-Based Design [3] which represent two major paradigms for design under uncertainty. It should be pointed out that the emphases of these two paradigms are different. Robust design is a method for improving the quality of a product through minimizing the effect of variations without eliminating the causes [4]. It emphasizes on achieving the robustness of system performance (for design objective). On the other hand, the reliability-based design approach focuses on maintaining design feasibility (for design constraints) at expected probabilistic levels. It is our belief that the needs of robustness and reliability should be integrated.

A common challenge that designers face when using either robust design or reliability-based design is the tradeoff between

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accuracy and efficiency. Specifically, under the robust design paradigm, mean and variance of system performance are evaluated for assessing the design objective and the probabilistic constraints are simplified using either the worst-case scenario (sensitivity analysis based) or the moment-matching formulation [4-6]. The commonly used method for evaluating the performance deviation (or variance) is the first order Taylor expansion. If the variances of random variables are large and the performance function is highly nonlinear, this approach may result in large errors. Though Monte Carlo simulation is generally more accurate, it is often unaffordable in engineering applications.

Under the reliability-based design paradigm, methods have been developed for efficiently assessing the probability of constraints being feasible (or called reliability). Many of these methods are based on the concept of the Most Probable Point [7-11], which emphasizes on assessing the tail performance of a probabilistic constraint. Deterministic objectives such as the performance at the mean values of random variables are often used to simplify the problem formulation. To overcome the difficulties (inefficiency) associated with double-loop procedures, sequential single-loop methods that separate the inner probabilistic assessment loop and the outer optimization loop have been proposed [12-15].

It is our belief that both robustness and reliability are the desired characteristics for design under uncertainty. Therefore these two paradigms need to be emerged in a unified probabilistic optimization formulation. Correspondingly efficient computational techniques need to be developed to facilitate the assessment of both characteristics in searching the probabilistic optimal solution. In this work, we propose an integrated framework for probabilistic optimization that can bring both the design objective robustness and the probabilistic design constraints into account. Two major developments are involved. The fundamental development is the employment of an inverse reliability strategy that uses percentile performance for assessing both the robustness objective and probabilistic constraints. The reliability requirements are formulated as inverse reliability constraints, which are assessed by equivalent percentile performances (inverse reliability formulation). The robustness is achieved through a design objective in which the variation of a design performance is evaluated through the percentile performance difference between the right and left tails of the performance. Corresponding to the use of percentile performance, the other major development is a new search algorithm of Most Probable Point of Inverse Reliability (MPPIR). The new algorithm can be used to efficiently evaluate the robustness and reliability in the proposed formulation. In the remaining part of this paper, we will demonstrate the benefits of the proposed integrated framework for probabilistic optimization and the effectiveness of the MPPIR search algorithm.

The paper is organized as follows. Some background information of probabilistic design optimization is presented in Section 2. The integrated framework for probabilistic optimization based on the inverse reliability strategy is introduced in Section 3. The MPPIR search method is described in Section 4. The verification of the MPPIR search algorithm and an automotive application – combustion engine piston slap design are given in Section 5. Section 6 is the closure of the paper.

2. A GENERAL PROBABILISTIC MODEL

A typical probabilistic design model is given by

Minimize: $f(\mathbf{d}, \mathbf{X}, \mathbf{P}) = v_{g_{obj}}$

Design Variable $DV = \{\mathbf{d}, \boldsymbol{\nu}_x\}$ (1)

Subject to: $\text{Prob}\{g_i(\mathbf{d}, \mathbf{X}, \mathbf{P}) \le 0\} \ge \alpha_i$, i= 1, 2, …, m

In the above model, f is the design objective, which is the probabilistic characteristic $v_{g_{obj}}$ of the objective function g_{obj}. $v_{g_{obj}}$ may be the mean, the standard deviation, or the combination of both. **d** is the vector of deterministic design variables or deterministic control factors. **X** is the vector of random design variables or called random control factors. **P** is the vector of random design parameters or noise factors. The difference between a design variable (either deterministic or random) and a design parameter is that the former is changeable and controllable while the latter is not. Taking a gear design as an example, the number of teeth on a gear is a deterministic design variable without any randomness associated with it. The radius and width of a gear are examples of random design variables due to the variations from the manufacturing process. Examples of random parameters include the uncontrollable variables, such as temperature, the external force, etc. The design variables in model (1) are **d** and the distribution parameters $\boldsymbol{\nu}_x$ of random design variables **X**. Examples of the distribution parameters $\boldsymbol{\nu}_x$ include the mean $\boldsymbol{\mu}_x$, the standard deviation $\boldsymbol{\sigma}_x$, etc. $g_i(\mathbf{d}, \mathbf{X}, \mathbf{P})$ (i=1, 2, …, m) are constraint functions; $\text{Prob}\{\cdot\}$ denotes a probability and α_i (i=1, 2, …, m) stand for desired probabilities of constraint satisfaction; m is the number of constraints. Note that both the objective performance g_{obj} and constraint performance g_i are system performances. In the remainder of this paper, we use g to denote a system performance that can be used either as an objective or a constraint.

In the above design model, the design feasibility is formulated as the probability of constraint satisfaction $g(\mathbf{d}, \mathbf{X}, \mathbf{P}) \le 0$ greater than or equal to a desired probability α. Usually we call this probability reliability. As shown in Fig. 1, the probability of $g(\mathbf{d}, \mathbf{X}, \mathbf{P}) \le 0$ is the area underneath the curve of probability density function (PDF) of g for $g \le 0$, and this area should be greater than or equal to α.

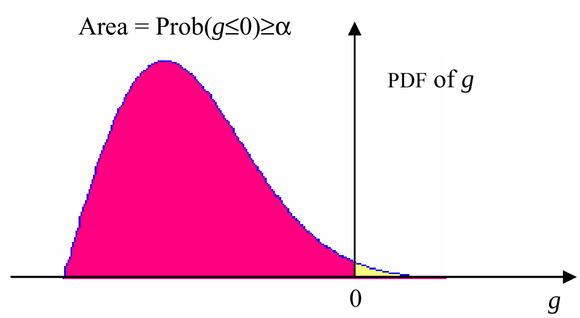

Figure 1. The Concept of Reliability

In a robust design, the robustness of a design objective can be achieved by simultaneously "optimizing the mean performance μ_{obj}" and "minimizing the performance variance σ_{obj}" [16]. Both random variables **X** and random parameters **P** could be the contributing sources of design variations. Consequently, the system performance $g_{obj} = g_{obj}(\mathbf{d}, \mathbf{X}, \mathbf{P})$ is a function of random variables. Its mean value μ_{obj} and variance σ^2_{obj} are to be minimized. The form of the objective can be expressed as

$$\min \quad f = \{\mu_{obj}, \sigma_{obj}\} \tag{2}$$

Different from robust design, the emphasis of the conventional reliability-based design is on maintaining the reliability of a constraint (design feasibility requirement). Usually only the nominal value is considered for the objective, which is calculated at the means of random variables, i.e.

$$\min \quad f = G_{obj} = g_{obj}(\mathbf{d}, \boldsymbol{\mu}_X, \boldsymbol{\mu}_P) \tag{3}$$

In this work, a unified probabilistic optimization formulation is used to integrate the robustness and reliability considerations. As shown in the following model, the robustness requirement is captured by the design objective while the reliability considerations are modeled using probabilistic constraints.

$$\begin{aligned} &\min \quad f = \{\mu_{obj}, \sigma_{obj}\} \\ &\text{Design Variable } DV = \{\mathbf{d}, \mathbf{v}_x\} \\ &\text{Subject to: } \text{Prob}\{g_i(\mathbf{d}, \mathbf{X}, \mathbf{P}) \le 0\} \ge \alpha_i, \ \text{i}= 1, 2, \ldots, m \end{aligned} \tag{4}$$

Model (4) represents a multiobjective optimization problem where the tradeoff needs to be made between minimizing the mean performance and minimizing the performance variation [16]. How to construct an objective function representing designer's preference in making the tradeoff is not the focus of this study. Here we assume that a single objective function is constructed based on both the mean and variance criteria.

3. AN INVERSE RELIABILITY STRATEGY FOR REFORMULATING THE PROBABILISTIC OPTIMIZATION MODEL

An inverse reliability strategy is proposed in this work to reformulate the probabilistic optimization formulation shown in model (4). This development is motivated by the need for developing computationally efficient techniques for solving the integrated probabilistic optimization model and the need for providing a more accurate assessment of performance dispersion in improving system robustness. In conventional reliability analysis, given a specific system performance, one is interested in finding the probability of the system performance greater or less than that specific performance. This is a typical reliability analysis [17]. With an inverse reliability problem, we will focus on finding a specific system performance that corresponds to a given reliability. This task is considered as solving an inverse reliability problem [18-19]. We employ inverse reliability formulations for assessing both the robustness objective and the probabilistic constraints. First we will discuss how to model the design feasibility (probabilistic constraint) using the inverse reliability strategy. We will then present a new inverse reliability formulation for assessing robustness of a design objective.

Modeling Design Feasibility by Inverse Reliability Strategy

Du and Chen [6] discussed the commonly used techniques for modeling design feasibility under uncertainty and they concluded that the ideal technique is the probabilistic formulation presented in model (4). However, to use model (4), we need to evaluate the reliability $\text{Prob}\{g_i(\mathbf{d}, \mathbf{X}, \mathbf{P}) \le 0\}$ for each probabilistic function $g_i(\mathbf{d}, \mathbf{X}, \mathbf{P})$. In probabilistic optimization with multiple constraints, some constraints may never be active and consequently their reliabilities are extremely high (approaching 1.0). Although these constraints are the least critical, the evaluations of these reliabilities will unfortunately dominate the computational effort in a probabilistic design process. The solution to this problem is to perform the reliability assessment only up to the necessary level. To this end, a formulation of percentile performance (inverse reliability) has been proposed to replace the reliability formulation [13-15, 20]. The percentile performance formulation is given by

$$g^\alpha \le 0, \tag{5}$$

where g^α is the α-percentile performance of $g(\mathbf{d}, \mathbf{X}, \mathbf{P})$, namely,

$$\text{Prob}\{g(\mathbf{d}, \mathbf{X}, \mathbf{P}) \le g^\alpha\} = \alpha \tag{6}$$

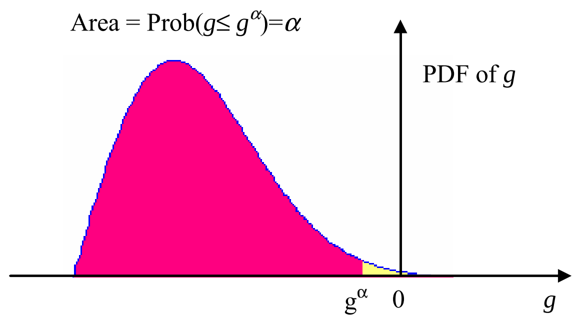

Figure 2. An α-Percentile of a Constraint Function

Eqn. (6) indicates that the probability of $g(\mathbf{d}, \mathbf{X}, \mathbf{P})$ less than or equal to α-percentile performance g^α is exactly equal to the desired reliability α. The concept is demonstrated in Fig. 2. If the shaded area, the probability in Eqn. (6), is equal to the desired reliability α, then the point g^α on g axis is called α-percentile value of function g. From Fig. 2 we see that, $g^\alpha \le 0$ indicates that $\text{Prob}\{g_i(\mathbf{d}, \mathbf{X}, \mathbf{P}) \le 0\} \ge \alpha$, which means that the probabilistic constraint is feasible. Therefore, the original constraints that require the reliability assessment are now converted to equivalent constraints that require evaluations of α-percentile performance. Instead of checking the actual reliability, the location of g^α will now determine the feasibility of a constraint. For simplicity, we use percentile performance to stand for α-percentile performance. It has also been shown that with percentile formulation we can avoid singularity problems which may occur in solving a direct reliability model (model (4)) during the iterative reliability assessment procedure [20].

Modeling Design Objective Robustness by Inverse Reliability Strategy

As mentioned in Section 1, the use of Taylor expansion is inaccurate in estimating the standard deviation of a performance. We may seek alternative methods that can accomplish the goal of shrinking the dispersion of a system performance in objective robustness.

With the original Taguchi's robust design method, to reduce the number of experiments, "compound noise" is used to represent conditions based on combinations of several noise factors that give high or low quality characteristics [2]. The robust design objective is then represented by the difference of system performance at two compound noise combinations corresponding to high or low quality characteristics respectively. The strategy of compound noise could significantly reduce the number of experiments (or number of simulations) since the performance is evaluated only at two points which supposedly correspond to the highest and lowest quality. However, certain conditions need to be satisfied to use this ad hoc approach [2]. For example, (1) we must know what the major noise factors are; (2) we must know the directionality of their effects on the system performance; and (3) the directionality of those effects of noise factors should not depend on the settings of the control factors. If the last two conditions are violated, the effect of one noise factor may get compensated by another noise factor and then the robust design based on the compound noise can give confusing and misleading results. Taguchi suggested a typical value of $\pm\sqrt{3/2}\sigma$ for noise levels, which is not always the most/least demanding direction and can lead to wrong conclusion in design selection.

To utilize the idea of compound noise but to overcome the aforementioned drawbacks, we propose to use percentile performance difference to represent the variation of a system performance. The percentile performance difference is given by

$$\Delta g_{obj\alpha_1}^{\alpha_2} = g_{obj}^{\alpha_2} - g_{obj}^{\alpha_1}, \qquad (7)$$

in which α_1 and α_2 are probability levels or the cumulative distribution functions (CDFs) of g given by

$$\text{Prob}\{f(\mathbf{d},\ \mathbf{X},\ \mathbf{P}) \le g_{obj}^{\alpha_i}\} = \alpha_i \quad (\text{i} = 1, 2) \qquad (8)$$

α_1 is a left-tail CDF, for example, 0.05 or 0.1, which represents the system performance at the left tail of its distribution and α_2 is a right-tail CDF, for example, 0.95 or 0.99. Percentile performances $g_{obj}^{\alpha_1}(\mathbf{d},\ \mathbf{X},\ \mathbf{P})$ and $g_{obj}^{\alpha_2}(\mathbf{d},\ \mathbf{X},\ \mathbf{P})$ represent high and low (or low and high) system quality respectively. From Fig. 3 we see that the percentile performance difference reflects the variation range of a system performance. Minimizing the mean of objective function shifts the location of the distribution towards left, while minimizing the percentile performance difference helps to shrink the range of the distribution.

As we will see later, the two percentile performances $g_{obj}^{\alpha_1}(\mathbf{d},\ \mathbf{X},\ \mathbf{P})$ and $g_{obj}^{\alpha_2}(\mathbf{d},\ \mathbf{X},\ \mathbf{P})$ can be evaluated at two MPP-based compound noise settings corresponding to the probabilities α_1 and α_2, respectively. Different from Taguchi's compound noise factor where only random parameters (noise factors) are involved, with our approach, a compound noise factor includes a combination of all random variables, namely, random control factors **X** and noise factors **P** (therefore we call a compound noise factor "compound noise setting" in subsequent sections). Both the effects of uncertainties of random control factors and noise factors are considered in Eqn. (7).

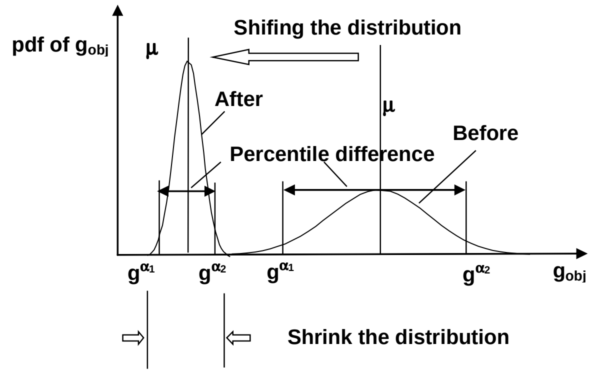

Figure 3. Percentile Difference for Shrinking the Distribution

There are several advantages of using percentile performance difference to replace the conventional performance variance (or standard deviation) for robustness assessment. One major advantage is that a percentile performance is related to the probability at the tail areas of a system performance distribution and therefore it carries more information than the standard deviation, for instance, the skewness of a distribution, while the standard deviation only captures the dispersion around the mean value. Also with percentile formulation, we can immediately know to what extent or at what confidence level the design robustness is achieved. This confidence level is given by $\alpha_2 - \alpha_1$. The other major advantage is related to the computational efficiency achieved by using inverse reliability assessment (percentile evaluation) for both robustness objective and probabilistic constraints [15]. As will be shown in the next section, the inverse reliability method allows us to calculate the percentile performance with high accuracy in an efficient manner. Combined with the concept of the Most Probable Point (MPP), the percentile formulation gives us a reasonable compound noise setting in robustness evaluation.

In summary, using the inverse reliability strategy, the unified probabilistic optimization model for integrated robustness and reliability design becomes

$$\begin{aligned} &\min \quad \{\mu_{G_{obj}},\ \Delta g_{obj\alpha_1}^{\alpha_2}\} \\ &\text{Design Variable } DV = \{\mathbf{d},\ \mathbf{v_x}\} \qquad (9) \\ &\text{Subject to: } g_i^{\alpha}(\mathbf{d},\ \mathbf{X},\ \mathbf{P}) \le 0,\ \text{i}=1, 2, \ldots, m \end{aligned}$$

In addition to the general model (9), for the smaller-the-better type robust design, the design objective can also be formulated as

$$\min \quad g_{obj}^{\alpha} \qquad (10)$$

in which α is a large probability, for example, 0.95 or 0.99. The objective function at the right tail of its distribution is to be minimized. Herein we have only one design objective and the design becomes a single objective problem.

For the larger-the-better type robust design, the design objective can also be formulated as

$$\max \quad g_{obj}^{\alpha} \tag{11}$$

in which α is a small probability, for example, 0.05 or 0.01. The objective function at the left tail of its distribution is to be maximized. We also have a single design objective in this case.

4. THE INVERSE RELIABILITY ASSESSMENT METHOD

To solve model (9) efficiently, an efficient search algorithm for the Most Probable Point of Inverse Reliability (MPPIR) is developed in this work to evaluate the percentile performances. We will discuss the proposed MPPIR search algorithm and then explain how the MPPIR method is related to the compound noise setting.

Background of MPP Search for the Inverse Reliability Problem

The MPP concept was originally developed in the structural reliability area [3] with the purpose of reliability assessment. With the MPP approach, the random variables $\mathbf{Y} = (\mathbf{X},\ \mathbf{P})$ are transformed into an independent and standardized normal space $\mathbf{U} = (\mathbf{U_X}, \mathbf{U_P})$. The transformation is given by [22],

$$U_i = \Phi^{-1}[F_{Yi}(Y_i)], \tag{12}$$

where Φ^{-1} is the inverse of a standard normal distribution and F is a CDF of a general random variable $\mathbf{Y}$.

The MPP is formally defined in the standardized normal space as the minimum distance point on the constraint boundary $g(\mathbf{d},\ \mathbf{X},\ \mathbf{P}) = g(\mathbf{d},\ \mathbf{U_X},\ \mathbf{U_P}) = 0$ to the origin. The minimum distance β is called reliability index. When the First Order Reliability Method (FORM) [21] is used, the reliability is given by

$$\alpha = \text{Prob}\{g(\mathbf{d},\ \mathbf{X},\ \mathbf{P}) \le 0\} = \Phi(\beta) \tag{13}$$

where Φ is the standard normal distribution function. Finding the MPP and the reliability index is a minimization problem, which usually involves an iterative search process. For details about the MPP based method, refer to [17].

In an inverse reliability problem, the required reliability α is given and the percentile performance corresponding to α is to be evaluated. Form Eqn. (13), the reliability index β is given by

$$\beta = \Phi^{-1}(\alpha) \tag{14}$$

Note that Eqn. (14) is applicable for $\alpha \ge 0.5$. When $\alpha < 0.5$, it becomes

$$\beta = \Phi^{-1}(1-\alpha). \tag{15}$$

As shown in Fig. 4, the MPPIR is the common point (tangent point) of a hyper sphere with radius β (β-sphere) in U-space and the contour of $g(\mathbf{U})$. At this point $g(\mathbf{U})$ reaches its minimum (or maximum). Whether the function is minimum or maximum at the MPPIR depends on to which tail the MPPIR corresponds. When the MPPIR corresponds to the left tail, we have a minimization problem otherwise we have a maximization problem. We discuss here the minimization only, but the same principle can be applied to a maximization problem. Fig. 4 shows the MPPIR where g(U) is to be minimized. An MPPIR problem is modeled as

$$\begin{cases} \min g(\mathbf{u}) \\ \text{subject to } (\mathbf{u}^T\mathbf{u})^{1/2} = \beta \end{cases} \tag{16}$$

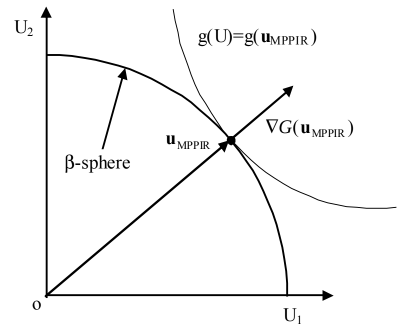

Figure 4. Inverse Most Probable Point

Once the MPPIR is identified, the percentile performance is calculated by

$$g^{\alpha} = g(\mathbf{u}_{\text{MPPIR}}) = g(\mathbf{x}_{\text{MPPIR}}), \tag{17}$$

which is the g function evaluated at the MPPIR.

Several existing methods can be used to solve model (16), including optimization techniques [23], traditional MPP search algorithms [17], Diagonal Direction Method [24], and the Hybrid Mean Value (HMV) Method [25]. The optimization method is a generic method but may not be efficient to solve the special type of minimization problem in model (16). Other methods cannot guarantee convergence. The solution found could be a saddle point or a maximum point instead. Most of the existing MPPIR search algorithms have convergence difficulties for non-concave and non-convex problems. It is our goal in this research to develop a new efficient MPPIR search algorithm that can be used for any types of system performance functions and is robust in its convergence behavior.

The Proposed MPPIR Search Algorithm

In developing an improved MPPIR search algorithm, we aim to improve the performance of the algorithm in two categories: 1) efficiency: to find the MPPIR with the number of function evaluations as small as possible for any regular (well-behaved) functions and 2) robustness: to avoid divergence caused by irregular limit state functions.

Demonstrated in Fig. 4 for a two dimensional case, the MPPIR is the tangent point of the β-sphere and the contour of function $g(\mathbf{U})$ in the U-space. At MPPIR, the vector $\mathbf{u}_{IMPP}$ connecting the MPPIR $\mathbf{u}_{IMPP}$ and the origin o should overlap with the negative gradient $-\nabla g(\mathbf{u}_{IMPP})$ of function $g(\mathbf{U})$.

The angle between $\mathbf{u}_k$ and $\nabla G(\mathbf{u}_k)$ is calculated by

$$\gamma_k = \cos^{-1} \frac{\mathbf{u}_k \cdot \nabla g(\mathbf{u}_k)}{\|\mathbf{u}_k\| \cdot \|\nabla g(\mathbf{u}_k)\|} \tag{18}$$

At the MPPIR, the angle γ_k should be zero. To satisfy this condition, the search process starts from the steepest descent direction and this is what a traditional MPP search algorithm does. When this direction leads to an increased function value due to the irregular function behavior, the second measure – an arc search procedure will be performed. It is called arc search because the search of the inverse MPP is along an arc on the β-sphere. The arc search can avoid converging to a maximum point or a saddle point. Note that for convenience, the search procedure is illustrated here in a two dimensional space. For higher dimensional problems, a plane, a curve, or a circle discussed for the two dimensional case will be a hyper plane, a hyper surface, or a hyper sphere, respectively.

Suppose the current point is $\mathbf{u}_{k-1}$ (k stands for the kth iteration in searching the MPPIR). At first, the steepest descent direction $-\nabla g(\mathbf{u}_{k-1})$ is used to obtain the new point $\mathbf{u}_k$ on the β-sphere by the following equation [24],

$$\mathbf{u}_k = -\beta \frac{\nabla g(\mathbf{u}_{k-1})}{\|\nabla g(\mathbf{u}_{k-1})\|} \tag{19}$$

Since the feature of the steepest descent direction of $-\nabla g(\mathbf{u}_{k-1})$ is valid locally around $\mathbf{u}_{k-1}$, there is a need to check the limit state function value to see whether there is a progress when using Eqn.(19). If $g(\mathbf{u}_k) < g(\mathbf{u}_{k-1})$, there indeed is a progress and the next iteration will follow Eqn. (19) again. If $g(\mathbf{u}_k) \geq g(\mathbf{u}_{k-1})$, it indicates that $\mathbf{u}_k$ is not improved compared with $\mathbf{u}_{k-1}$. An arc search will then be performed to identify a new $\mathbf{u}_k$ that leads to a decreased function value.

As shown in Fig. 5, the arc search is to find the minimum function value point on the intersection of β-sphere and the plane determined by the vectors $\mathbf{u}_k$ and $-\nabla g(\mathbf{u}_k)$. Apparently, the plane passes through the origin o and the search path is an arc of the β-sphere.

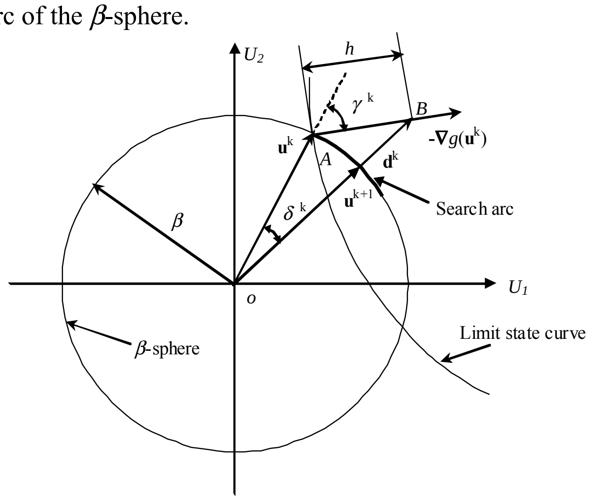

Figure 5. New Search Direction $\mathbf{d}^k$ in An Arc Search

To find the next point $\mathbf{u}_{k+1}$, a search direction $\mathbf{d}_k$ is identified as the linear combination of $\mathbf{u}_k$ and $-\nabla g(\mathbf{u}_k)$:

$$\mathbf{d}_k = \mathbf{u}_k - h\nabla g(\mathbf{u}_k), \tag{20}$$

where h is given by

$$h = \frac{\beta \sin \delta^k}{\sin(\gamma^k - \delta^k)}. \tag{21}$$

Based on Eqns. (20) and (21), we obtain

$$\mathbf{d}_k = \mathbf{u}_k - \frac{\beta \sin \delta^k}{\sin(\gamma^k - \delta^k)} \nabla g(\mathbf{u}_k) \tag{22}$$

Hence, $\mathbf{u}_{k+1}$ is calculated by

$$\mathbf{u}_{k+1} = \beta \frac{\mathbf{d}_k}{\|\mathbf{d}_k\|} = \beta \frac{\mathbf{u}_k - \frac{\beta \sin \delta^k}{\sin(\gamma^k - \delta^k)} \nabla g(\mathbf{u}_k)}{\left\| \mathbf{u}_k - \frac{\beta \sin \delta^k}{\sin(\gamma^k - \delta^k)} \nabla g(\mathbf{u}_k) \right\|} \tag{23}$$

The arc search is then formulated as a one-dimensional minimization problem represented by

$$\begin{cases} \mathit{Find}\text{: The angle } \delta^k \\ \mathit{Minimize}\ \ g(\mathbf{u}_{k+1}) = g\left(\beta \dfrac{\mathbf{u}_k - \frac{\beta \sin \delta^k}{\sin(\gamma^k - \delta^k)} \nabla g(\mathbf{u}_k)}{\left\| \mathbf{u}_k - \frac{\beta \sin \delta^k}{\sin(\gamma^k - \delta^k)} \nabla g(\mathbf{u}_k) \right\|} \right) \end{cases} \tag{24}$$

To further illustrate the procedure of an arc search, the progress of the proposed method is demonstrated in Fig. 6 for a three-dimensional problem in the case that all the points are obtained by the arc search. Suppose the current point is $\mathbf{u}_1$, the new point $\mathbf{u}_2$ is determined in the plane by the vector $\mathbf{u}_1$ and vector $-\nabla g(\mathbf{u}_1)$. Geometrically, this new point is the tangent point of the projection of $-\nabla g(\mathbf{u}_1)$ on β-sphere (an arc segment) and the projection of $g(\mathbf{u}_1)$ on β-sphere (contours on β-sphere). Obviously, the value of $g(\mathbf{U})$ at $\mathbf{u}_2$ is smaller than the one at $\mathbf{u}_1$. Analogously, the same procedure is conducted to find points u_3, u_4, $\cdots$ until the vector $\mathbf{u}_i$ overlaps with the vector $-\nabla g(\mathbf{u}_i)$.

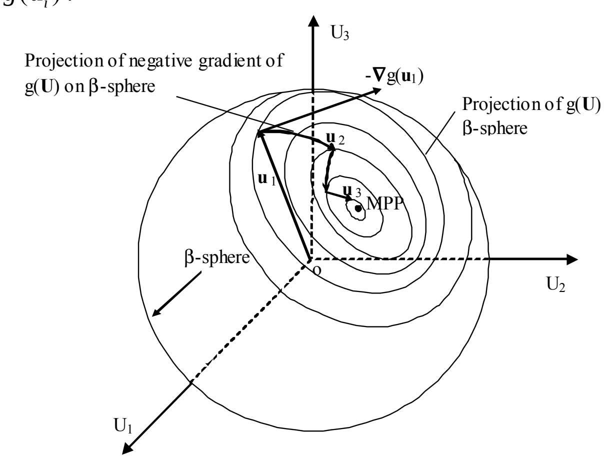

Figure 6. An Arc Search

If we let the current point be $\mathbf{u}_k$ on the β-sphere, the search process is summarized as follows:

1) Calculate the gradient $\nabla g(\mathbf{u}_k)$ at $\mathbf{u}_k$.
2) Calculate the angle γ_k between $-\nabla g(\mathbf{u}_k)$ and $\mathbf{u}_k$ using Eqn. (18).
3) If $\gamma_k \leq \varepsilon$, $\mathbf{u}_k$ is the MPPIR and go to 4), otherwise, go to 1). ε is a small angle, for example, 0.1°.
4) Calculate the percentile performance $g(\mathbf{u}_k)$ and stop;
5) If $g(\mathbf{u}_k) < g(\mathbf{u}_{k-1})$, update the point by $\mathbf{u}_{k+1} = -\beta \frac{\nabla g(\mathbf{u}_k)}{|\nabla g(\mathbf{u}_k)|}$, and $k = k+1$, then go to 1); otherwise, use Eqns. (20) ~ (24) to perform the arc

search to locate the new point $\mathbf{u}_{k+1}$ and update k by $k = k+1$. Then, go to 1).

To make the search process robust and efficient, the adaptive step size is also employed for the finite difference derivative evaluations.

For a "well-behaved" limit state function, for example, a convex limit state function, the steepest descent direction method works well and function g drops down constantly. In this case the efficiency of the proposed method is as good as other existing methods. When function g is convex, or non-concave and non-convex, the arc search in the proposed method guarantees the descent of the limit state function and therefore the convergence. Hence the method is robust to various types of limit-state functions. We will further verify it through comparative studies.

Verifications of the MPPIR Search Algorithm

For the purpose of verification, many testing problems with different types of limit state behaviors have been used to evaluate the proposed MPPIR algorithm [26]. The tested functional behaviors range from convex, concave, to non-convex and non-concave functions. The results show that our method is robust and efficient. By robust we mean that the algorithm performs well for different functional behaviors. By efficient we mean that the number of function evaluations taken by the algorithm is small. We will not present all the test problems in this paper. One example of non-convex and non-concave function is given below.

The function is given by

$$g(\mathbf{X}) = 4-(X_1+0.25)^2+(X_1+0.25)^3+(X_1+0.25)^4-X_2 \quad (25)$$

where $X_1 \sim N(0.0,\ 1.0)$ and $X_2 \sim N(0.0,\ 1.0)$; $N(\mu,\ \sigma)$ stands for a normal distribution with mean μ and standard deviation σ. The radius of the β-sphere is β=3.0, which corresponds to reliability α=0.9987 based on FORM.

From Table 1 we see that the proposed MPPIR method succeeds in finding the most probable point for β=3.0 while the traditional method does not. To confirm the accuracy, we also employ several commonly used optimization algorithms – the modified feasible direction (MFD) method, sequential quadratic programming (SQP), sequential linear programming (SLP), and the optimization software LINGO to solve the MPPIR search as an optimization problem. We found that the MFD method produces a similar result as the proposed MPPIR method. The SQP, SLP and LINGO find different MPPIR solutions from those of the proposed MPPIR method and the MFD method. If we look at function values, we can conclude that the proposed MPPIR method and the MFD method give better results since the function is better minimized than other methods. We provide in Table 1 the number of function evaluations used by each method.

Based on the working principle and the problems tested, we conclude that the proposed method has the same efficiency as the traditional methods if a limit state function is convex. When the limit state function is concave or non-convex and non-concave, the proposed method is still efficient and can avoid divergence.

Table 1. Comparison of the Inverse MPP Search Results

Method		$g(\mathbf{x}_{MPPIR})$	$\mathbf{u}_{MPPIR}$	N
Traditional Method		Diverge		
Proposed Method		0.2440	(-1.3503, 2.6789)	20
Optimization algorithms	MFD	0.2433	(-1.3505, 2.6795)	89
	SQP	0.9332	(-1.527, 2.9962)	28
	SLP	0.9330	(-1.5547, 2.9963)	45
	Lingo	0.9333	(0.1552, 2.9960)	-

N - Number of Function Evaluations

MPPIR-Based Compound Noise Setting

Here we provide some further discussions on why MPPIR benefits the robustness assessment. As we have discussed in Section 3, to obtain the effect of noise factors in robust design, Taguchi used the compound noise strategy [2] with which the compound noise factors are combined into two extreme levels of compound noises: the least and most demanding respectively. If we plot Tachuchi's compound noises for a two-dimensional problem in the standard normal space, the two points of compound noises are the two end points of a line segment passing the origin and making an angle of 45° or 135° with respect to u_1 axis. As we have discussed in Section 3.2, with Taguchi's strategy, the constant orientation (45° or 135°) of the compound noise indicates constant directionality of effects of random variables and may lead to wrong search direction to performance robustness. The reason is that the compound noise points are "the worst case" combinations for only the random variables themselves but no sensitivity information of system performance with respect to random variables is considered. On the other hand, since the MPPIR is the most likelihood point of the system performance corresponding to the least or the most demand, it is natural to consider an MPPIR as a noise compound setting. An MPPIR is not necessarily a "the worst case" combination of the random variables, rather it is the "most probable" combination of the random variables that represent the most or the least demanding performance (see Fig. 7).

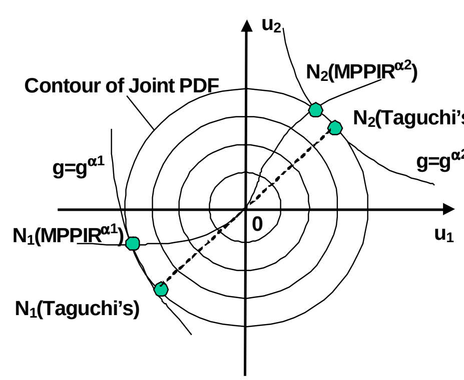

Figure 7. MPP-Based Compound Noise Setting

When the design point (composed of deterministic control factors and means of random control factors) changes in the design space, the location of an MPPIR will change accordingly. Therefore, the MPPIR based compound noise overcomes the drawbacks of the original compound noise discussed in Section 3.2. In the next section, a design example is used to demonstrate the advantage of using the MPPIR based compound noise for evaluating design robustness.

5. APPLICATION – DESIGN OF COMBUSTION ENGINE PISTON SLAP UNDER UNCERTAINTY

In this section, we use a combustion engine piston slap design to demonstrate the use of our proposed percentile performance based probabilistic design formulation along with the MPPIR search method presented in Section 4.

In vehicle design, total vehicle customer satisfaction is strongly linked to the level of satisfaction a customer has with the vehicle's engine. One of the key elements of customer satisfaction is the Noise, Vibration and Harshness (NVH) characteristics. Piston slap is an unwanted engine noise that is the result of piston secondary motion, namely, the departure of the piston from the nominal motion prescribed by the slider crank mechanism. This secondary motion is caused by a combination of transient forces and moments acting on the piston during engine operation and the presence of clearances between the piston and the cylinder liner [27]. This combination results in both a lateral movement of the piston within the cylinder and a rotation of the piston about the piston pin, and it causes the piston to impact the cylinder wall at regular intervals. These impacts may result in the objectionable engine noise known as piston slap.

A dynamic power cylinder model is developed to predict piston motion and side loads within the cylinder [28]. This correlated model is the basis of a comprehensive analytical design of experiments where both piston noise and piston friction are monitored. The results of the design of experiments (DOE) are used to generate surrogate models for piston friction and for piston noise. The integrated robust and reliability design model proposed in this work is used to make design decisions.

There are six random variables in this design, including four control factors (random design variables) $\mathbf{X}=\{X_1, X_2, X_3, X_4\}$ which are the skirt length, skirt profile, skirt ovality, and pin offset. There are two noise factors $\mathbf{P}=\{P_1, P_2\}$ which are clearances between the piston and the cylinder liner and the location of peak pressure. No deterministic design variables **d** exist in this application. Descriptions of random variables are listed in Table 2.

Table 2. Information of Random Variables

Variable	Mean	Standard deviation	Distribution
X_1	v_1	0.01	Normal
X_2	v_2	0.0001	Normal
X_3	v_3	0.0001	Normal
X_4	v_4	0.01	Normal
P_1	50.0	11.67	Normal
P_2	17.0	0.83	Normal

The robust design objective is to minimize the noise (g_{obj}) and its variation. The friction (g_1) is considered as a reliability design constraint such that the probability of friction g_1 less than 7 (N) should be greater than 0.99. Using the proposed inverse reliability strategy (percentile performance formulation for both objective and constraints), we create the following design model

$$\min \quad w_1 \frac{\mu_{obj}}{\mu^*_{obj}} + w_2 \frac{\Delta g_{obj\,0.05}^{\;0.95}}{\Delta^* g_{obj\,0.05}^{\;0.95}}$$

$$DV = \{ \boldsymbol{\mu}_{\mathbf{x}} \} \qquad (25)$$

$$\text{Subject to: } g_1^{0.99} - 7 \le 0$$

where w_1 and w_2 are weighting factors, μ^*_{obj} (obtained by $w_1 = 1$ and $w_2 = 0$) and $\Delta^* g_{obj\,0.01}^{\;0.99}$ (obtained by $w_1 = 0$ and $w_2 = 1$) are the ideal solutions used to normalize the two aspects in the objective, i.e., minimizing the mean performance and minimizing performance variation. The percentile levels for the objective are chosen at 99% and 1%, respectively.

For comparison, we also solve the problem using the traditional robust design objective and probabilistic constraint shown in the model below

$$\min \quad w_1 \frac{\mu_{obj}}{\mu^*_{obj}} + w_2 \frac{\sigma_{obj}}{\sigma^*_{obj}}$$

$$DV = \{ \boldsymbol{\mu}_{\mathbf{x}} \} \qquad (26)$$

$$\text{Subject to: } \text{Prob}\{g_1 - 7 \le 0\} \ge 0.99$$

in which the variation of the objective is represented by the standard deviation of the objective and the constraint is formulated using the original probabilistic constraint.

The Sequential Quadratic Programming is used as the optimization search algorithm to solve both models (25) and (26) and the proposed MPPIR algorithm is used to solve the percentile performance in model (25). The optimal solution to model (25) is given in Table 3.

Table 3. Optimal Solution from the Proposed Method

	Baseline	Optimal Point
Control Factor μ_x	{25.0, 2.0, 2.0, 1.0}	{22.32,3.0, 3.0, 1.21}
Mean of noise	54.03	53.77
99% percentile noise	60.61	56.84
1% percentile noise	51.47	53.39
99% and 1% percentile difference noise	9.14	3.45
99% percentile friction	1.88	0.0

From the result we see that after using the proposed method, the mean noise decreases to 53.77 dB from 54.03 dB at the baseline. In terms of robustness, the percentile difference of noise decreases significantly, from 9.14 dB to 3.45 dB. The optimal solution is the result of shifting the distribution of the engine noise leftward and shrinking its distribution range. It is noted that since we use 99% and 1% percentiles, we could say that we have achieved the robustness (3.45 dB percentile difference) at probability level of 0.98 (99%-1%). At the baseline, the 99% percentile of the friction is 1.88 (N). Since this percentile value is greater than zero, it indicates that the reliability of the constraint feasibility is less than the required reliability. Hence the design at the baseline is not a feasible design. With the proposed optimization model, the 99% percentile of the friction becomes 0.0 at the optimal point, which means that the reliability of the constraint feasibility is exactly equal to the required reliability. Hence we have achieved both the robustness and reliability for the piston design.

Table 4 shows the MPPIRs (compound noise setting) of the objective function (noise) at 95th and 5th percentile at the optimal point, respectively.

Table 4. MPPIRs at Baseline and the Optimal Point

Baseline	1% percentile			99% percentile		
	$\mathbf{u}_{mpp}$	$\mathbf{x}_{mpp}$	Direc	$\mathbf{u}_{mpp}$	$\mathbf{x}_{mpp}$	Direc
X_1	-0.0291	25.0	–	0.0114	25.0	+
X_2	-0.0013	2.0	–	0.0	2.0	+
X_3	0.0021	2.0	+	-0.0001	2.0	–
X_4	-0.2086	0.9979	–	-0.0389	0.9996	–
P_1	-2.3167	2.2972	–	2.2968	76.7960	+
P_2	0.024	17.02	+	0.3672	17.3060	–
Optimal Point	1% percentile			99% percentile		
	$\mathbf{u}_{mpp}$	$\mathbf{x}_{mpp}$	Direc	$\mathbf{u}_{mpp}$	$\mathbf{x}_{mpp}$	Direc
X_1	0.8776	22.3320	+	-0.0085	22.3230	+
X_2	0.0099	3.0	+	-0.0002	3.0	+
X_3	-0.0357	3.0	–	0.0002	3.0	–
X_4	-1.6044	1.1942	–	-0.0306	1.210	–
P_1	-0.2824	46.7050	–	2.1142	74.6660	+
P_2	-1.4095	15.8250	–	0.9701	17.8080	–

Direc – Directionality

An MPPIR gives a most probable combination of all the random variables at a specific percentile level. If the sign of a component of an MPPIR in U space is positive, i.e., the component of MPPIR in the original space is greater than its mean, the sign of its directionality is "+", and vice versa. From example, at the 1% percentile (the left trail of engine noise distribution), the directionality of X_1 being "-" means that the x_{1MPP} is at the right side of mean of X_1. If we increase the mean of X_1, we could reduce the probability (or the percentile level) if other parameters are fixed. Or in other words, we could shrink the distribution of the engine noise by increasing the mean of X_1. Similarly, if we decrease the mean of a random variable that has "+" directionality, we could also shrink the distribution of the engine noise. Now let us consider the right tail of the engine noise distribution at 99% percentile, if we decrease the mean of a random variable that has "-" directionality or increase the mean of a random variables that has "+" directionality, the probability (or percentile level) will increase and as a result, the dispersion of the engine noise distribution will be smaller. Adjusting the location of the MPPIR to achieve the robustness objective is carried out automatically when using the percentile performance formulation in the proposed optimization model.

It is observed that the directionality of the random variables may change in the process of the optimization. For instance, at the 1% percentile level, X_1 has "-" directionality at the baseline and "+" directionality at the optimal point. This indicates the nonlinearity of the problem. For nonlinear problems, traditional compound noise combinations [2] might give us misleading results due to its fixed directionality.

The traditional model (26) generated the similar optimal result, but the number of function evaluations is about six times higher than that of the proposed inverse reliability model (25). The savings mainly came from less computational effort required in the optimization model when using the inverse reliability formulation, where only the performance corresponding to the required reliability is evaluated and no actual reliability evaluation is needed. The MPPIR algorithm also plays a very important role in this saving since it provides an efficient tool for locating the MPPIR. From this example, we see that solving the proposed inverse reliability (percentile performance) based model is more efficient than solving a traditional probabilistic model.

6. CONCLUDING REMARKS

The central idea of the proposed inverse reliability strategy is to convert a traditional probabilistic optimization design model into an equivalent percentile performance based formulation. A uniqueness of this work is to apply this strategy to the integrated robust and reliability-based design problems. Under this strategy, the standard deviation of an objective performance is replaced by its percentile performance difference, and the reliability of a probabilistic constraint is replaced by its percentile performance corresponding to the required reliability. The same strategy can be extended to general probabilistic optimization problems where the objective represents any probabilistic characteristics that can be measured by multiple percentile performance. There are several advantages of using the proposed unified probabilistic design model. First, with the percentile formulations for both objective and constraints, the existing reliability methods are extended to the evaluation of the probabilistic characteristics of an objective in addition to that of the constraints. Second, the percentile formulation provides us more accurate evaluation of the variation of an objective performance compared to using the standard deviation in a traditional formulation. It also provides the probabilistic measurement of the robustness of an objective performance. Third, with the proposed formulation, we may obtain more reasonable compound noise combinations for a robust design compared to using the traditional approach proposed by Taguchi. Fourth, compared to solving a traditional probabilistic model, the proposed formulation is more efficient since it only needs to evaluate the constraint functions at the required reliability levels.

Along with the proposed unified probabilistic model, an efficient and robust MPPIR search algorithm is also developed in this work to support the inverse reliability strategy. Based on its working principle and the problems tested, we illustrate that the proposed MPPIR search algorithm is robust in its convergence performance for various types of system performance functions with either concave, or non-convex and non-concave behaviors. When the system performance function is convex, the efficiency of our proposed algorithm is at least as good as other existing MPPIR search algorithms.

In the examples shown in this paper, the inverse MPP search is based on the First Order Reliability Method (FORM). The accuracy is sufficient for a general probabilistic design. When higher accuracy is desired, the proposed method can be easily modified to accommodate the Second Order Reliability Method (SORM) [17]. The procedure is as follows: for a given required reliability α, using $\beta = \Phi^{-1}(\alpha)$ to find the radius of the β-sphere; locate the MPPIR using the MPPIR search algorithm and calculate the reliability by SORM; if the reliability is not equal to the required reliability α, change the radius β accordingly; repeat the procedure until the reliability calculated is equal to the required reliability. Then, evaluate the percentile at the MPPIR found in the last iteration.

In our current implementation, a double loop procedure that involves the inner probabilistic assessment loop and the outer optimization loop is used. We have developed a single-loop procedure [15] for reliability-based design where the objective is deterministic. The inverse reliability strategy gives a good staring point to seek an efficient single loop method for a general probabilistic optimization problem where the objective is not necessary deterministic.

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