Proceedings of DETC'03
ASME 2003 Design Engineering Technical Conferences and Computers and Information in Engineering Conference
Chicago, Illinois USA, September 2-6, 2003

DETC2003/DAC-48710

APPLICATION OF THE SEQUENTIAL OPTIMIZATION AND RELIABILITY ASSESSMENT METHOD TO STRUCTURAL DESIGN PROBLEMS

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ABSTRACT

The use of probabilistic optimization in structural design applications is hindered by the huge computational cost associated with evaluating probabilistic characteristics, where the computationally expensive finite element method (FEM) is often used for simulating design performance. In this paper, a Sequential Optimization and Reliability Assessment (SORA) method with analytical derivatives is applied to improve the efficiency of probabilistic structural optimization. With the SORA method, a single loop strategy that decouples the optimization and the reliability assessment is used to significantly reduce the computational demand of probabilistic optimization. Analytical sensitivities of displacement and stress functionals derived from finite element formulations are incorporated into the probability analysis without recurring excessive cost. The benefits of our proposed methods are demonstrated through two truss design problems by comparing the results with using conventional approaches. Results show that the SORA method with analytical derivatives is the most efficient with satisfactory accuracy.

KEY WORDS

probabilistic optimization, structural design, Sequential Optimization and Reliability Assessment, finite element method, analytical derivative

1. INTRODUCTION

The presence of uncertainty in the analysis and design has been recognized in the engineering community. The most challenging issue of a probabilistic design is the expense of computation associated with probabilistic analysis, i.e., the evaluation of probabilistic functions and their derivatives [1-4]. In probabilistic analysis, the system performance needs to be evaluated repeatedly, to capture its probabilistic characteristics. If the performance evaluation is carried out by the finite element method (FEM), such as in probabilistic structural design optimization, the computational burden may become unaffordable.

A lot of efforts have been made to improve the efficiency of probabilistic analysis. One critical probabilistic character that takes the most of the computational effort in probabilistic optimization is the *reliability*, i.e., the probability of a design constraint being feasible. The majority of the stochastic finite element methods (SFEM) [5-10] are basically a combination of deterministic FEM and probabilistic methods, such as the implementation of the Monte Carlo simulations (MCS) [11-16], and the employment of reliability index approaches based on the Most Probable Point (MPP) concept [17,18]. In structural design, reliability assessments are most concerned with improbable catastrophic events, i.e., probability at the tails of probability distribution functions. The computational expense of using sampling approaches is prohibitive in this case. The first order mean value (MVFO) methods are generally not sufficiently accurate [19-22]. The reliability index method, which uses approximations at a most probable point (MPP) of failure on a limit state surface [17,18], is the most promising technique since it is more affordable than the sampling methods and much more accurate than the mean-value methods. The accuracy of different probabilistic analysis methods is discussed in detail by Fox and Reh [23] and Thacker, et al. [24].

Even though the MPP based reliability index method can be used to improve the efficiency of probabilistic analysis, its use in probabilistic optimization is still very limited, especially for those problems with a large number of probabilistic

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constraints. Different constraints form different limit state functions for which the MPPs need to be searched separately, which becomes computationally cumbersome. There is a need for optimization strategies that can fundamentally improve the efficiency of probabilistic optimization. Two major categories of probabilistic optimization strategies exist in the literature. They are the *double-loop method* [2,25-27] and the *single-loop method* [4,28-31]. With the double-loop method, the loop of reliability analysis (an iterative procedure) is nested within the loop of optimization. At each iteration of optimization, multiple reliability analyses are performed for evaluating each probabilistic constraint and its derivatives, which is very time-consuming.

The single-loop method is aimed for improving the efficiency. With the single-loop method, the reliability analysis is included in the same loop as optimization instead of nested within. There are several single-loop strategies. One is called the "double-design-variable" method or the "single-loop-double-vector" approach [29]. Double-vector means that there are two vectors of design variables. One is the vector of mean values of the original design variables; the other contains the MPP values. This method will increase the dimension of a design problem, especially for those problems with many failure modes. Chen and Hasselman [30] proposed a "single-loop single-vector" approach which is derived in a normalized parameter space other than the standardized normal space (U space). It avoids the evaluation of the actual reliability index. But it is necessary to evaluate all constraint derivatives to identify all potentially active constraints at the beginning of the design, which may increase the computation effort. The accuracy of the Chen and Hasselman's method needs further investigation. Sues and Cesare [31] developed a single-loop method in which MPPs are computed before the first execution of the optimization and updated after each optimization iteration using the updated design variables. The accuracy of MPPs updated using their strategy may not be satisfactory. We recently developed the Sequential Optimization and Reliability Assessment (SORA) method [4], a single-loop method that converts a probabilistic design problem to an equivalent deterministic optimization problem followed by the inverse reliability assessment for checking the constraint feasibility in each cycle of probabilistic optimization. Our preliminary studies illustrate that the SORA method can significantly improve the efficiency of a probabilistic optimization. In this work, we apply the SORA method to probabilistic structural design optimization and further test its applicability to this category of applications.

In addition to improving the MPP search algorithm [32-35] and the probabilistic optimization strategy, improving the efficiency of derivative calculations will also contribute greatly to improving the efficiency of the entire probabilistic structural design optimization. The MPP search in reliability analysis needs to evaluate derivatives of each limit state function with respect to each random variable. Romero [36] argued that accurate finite-difference derivatives are not cost effective to obtain. The computation burden becomes even heavier if functions are evaluated by FEM. Probabilistic finite element (PFE) analysis [37,38] has been used in structure engineering to estimate the effects of input parameter uncertainty. Several analytical methods for sensitivity calculation in PFE were reviewed and compared by Guan and Melchers [39]. However, those methods usually are either not efficient or difficult to implement in probabilistic design. Another possible approach is the Automatic Differentiation (AD) method that calculates derivatives using computer-generated programs [40]. Since AD is based on the chain rule, it cannot take advantages of any possible physical insights. Furthermore, the programs generated by AD are often cumbersome and slow to execute. In this work, we propose to improve the computational efficiency of probabilistic structural design problems from two aspects. First we apply the Sequential Optimization and Reliability Assessment (SORA) method [4] to probabilistic optimization. Second, we derive and incorporate the analytical sensitivities of different functionals into the probability analysis without recurring excessive computational cost. The parameter uncertainties are directly included into the underlying FEM constitutive equations to obtain analytical formulation of derivatives for general truss design problems. The benefits of the proposed methods are demonstrated through two truss design problems by comparing the results with using conventional approaches.

The paper is organized as follows. In Section 2, a general probabilistic optimization model and the ideas behind the SORA method are introduced. In Section 3, the analytical derivations of the stress and displacement in FEM based truss design are provided. In Section 4, four different reliability-based design formulations are applied to two examples, the 5-bar and the 10-bar trusses design problems. The four formulations include: 1) double-loop strategy with probabilistic formulation; 2) double-loop strategy with percentile formulation; 3) SORA method using finite-difference for derivative calculation; 4) SORA method using analytical formulation for derivative calculation. The optimal solutions and efficiencies are compared. Conclusions are drawn in Section 5.

2. THE PROPOSED PROBABILISTIC OPTIMIZATION METHOD

2.1 Background Of Probabilistic Optimization

In probabilistic design, variations could be associated with both design variables and parameters. There are in total three categories of variables: **d** is a vector of deterministic design variables, **x** is a vector of random design variables, and **p** is a vector of random parameters. A typical probabilistic design model [35] is defined as:

$$\begin{aligned}&\textit{Find}\quad \mathbf{d},\ \boldsymbol{\mu}_x\\&\textit{Minimize}\quad f(\mathbf{d},\ \mathbf{x},\ \mathbf{p})\\&\textit{subject to}\quad P\{g_i(\mathbf{d},\mathbf{x},\mathbf{p})\ge 0\}\ge R_i,\ \ i=1,2,3,...,m.\end{aligned}\tag{1}$$

In probabilistic optimization, design feasibility, or called reliability, is defined as the probability of constraint satisfaction, P_i, for $g_i \ge 0$ should be no less than the required reliability level R_i.

2.2 The Concept Of The Most Probable Point (MPP)

The concept of the MPP is first introduced in structural reliability analysis [41] to approximate the reliability, shown as $P\{g_i(\mathbf{d},\mathbf{x},\mathbf{p}) \geq 0\}$ in model (1). With the transformation of both random design variables **x** and random parameters **p** into a standardized normal space, called U space, the Most Probable Point is formally defined in that space as the minimum distance point on the limit state function from which to the origin. If the limit state of function g is linear in the U space, its reliability is

$$P\{g(\mathbf{d},\mathbf{x},\mathbf{p}) \geq 0\} = P\{g(\mathbf{d},\mathbf{u}) \geq 0\} = \Phi(\beta). \quad (2)$$

Equation (2) is still a good approximation when the degree of nonlinearity of g is not high. Otherwise, higher order approximations, such as the Second Order Reliability Method (SORM) [42-44] should be applied. When there is correlation between random variables, the reduced variables should be transformed into an uncorrelated space first. Random variables following non-normal distributions should be transformed to equivalent independent standard normal variables and the equivalent mean and standard deviation can be estimated using the *Rackwitz-Fiessler* method, or *Chen-Lind* method [18], etc.

The problem of determining the value of reliability index β is equivalent to locating the MPP on the limit state surface. Such locating procedure is called the MPP search, which can be stated as an optimization problem that usually involves an iterative search process.

2.3 Difficulties With The Double-Loop Strategy

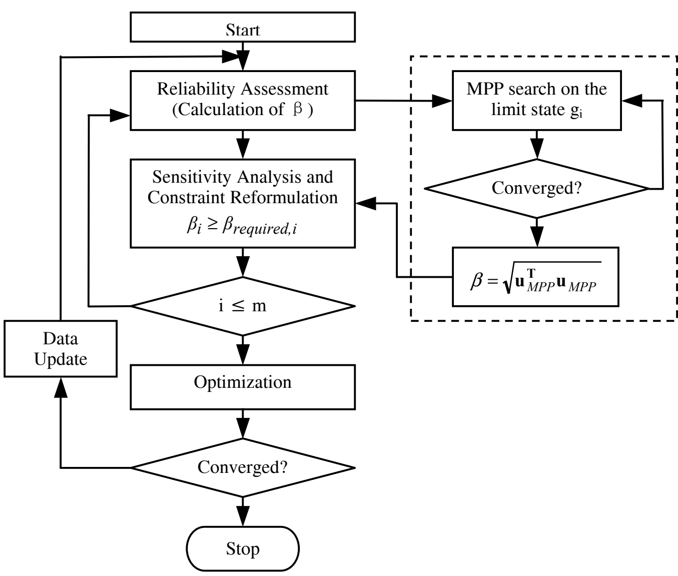

Figure 1. Flowchart of the Double-Loop Strategy

Since the reliability assessment is an iterative procedure, solving the probabilistic optimization model following the conventional optimization search strategy is referred to as the "double-loop" method, or called "double-loop-single-vector" approach by Chen and Hasselman [30]. The outer loop is for controlling an optimization search process; the inner loop is the reliability assessment. The flowchart of a double-loop based probabilistic design is shown in Fig. 1. The outer loop optimizer calls the inner loop reliability assessment repeatedly for gradient or function assessments. For each reliability assessment, there will be multiple evaluations of constraint functions. The reliability analysis needs to be performed for all probabilistic constraints. Therefore, the efficiency of double-loop methods is especially low when there are many failure modes.

2.4 The SORA Method

The SORA method is recently developed by us to improve the computational efficiency of reliability-based optimization through decoupling reliability assessment from optimization [4]. The SORA method transforms the model in Eq. (1) into a set of sequential equivalent deterministic optimization problems, followed by the confirmations of reliability. From cycle to cycle, the deterministic optimization formulation in a new cycle is updated based on the information of the MPP obtained from the reliability assessment in the previous cycle. This process is repeated until convergence. Three major ideas are introduced in the SORA method:

(1) Use R-percentile formulation to evaluate design feasibility only at the desired reliability level (R),

(2) Employ equivalent deterministic optimization and single loop strategy to reduce the number of reliability assessments,

(3) Use efficient MPP search algorithm (the MAMV-modified advance mean value method) for reliability assessment

The use of the R-percentile formulation instead of the original reliability assessment is based on the observation that the higher the actual reliability is, the more the computational effort is needed. For a probabilistic constraint, our concern is not to find the actual reliability of a limit state function, but to determine whether it is probabilistic feasible. Some probabilistic constraints may never be active whose reliability is very close to one. With the R-percentile formulation, instead of finding the actual reliability of a limit state, reliability assessment is performed at the desired level R_i to improve the efficiency.

Let the probability of constraint satisfaction equal to the required reliability (or percentile level) R, i.e., $P\{g(\mathbf{d},\mathbf{x},\mathbf{p}) \geq g^R\} = R$. The constant g^R is called the percentile performance of g. In Fig. 2, if the shaded area is equal to the desired reliability R, then the left ending point g^R on the g axis is called the R–percentile performance of function g. If g^R is no less than zero, the reliability of the limit state is no less than the required reliability, and therefore the constraint is feasible. The original probabilistic constraint $P\{g(\mathbf{d},\mathbf{x},\mathbf{p}) \geq 0\} \geq R$ can be rewritten as the following equivalent form

$$g^R \geq 0. \quad (3)$$

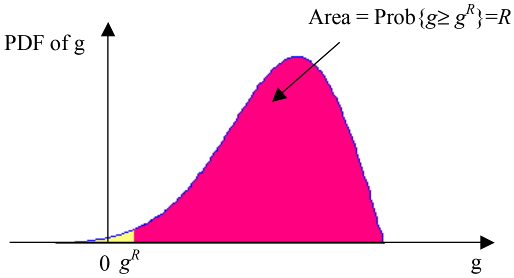

Figure 2. R-Percentile Performance of A Constraint Function [4]

As the function of the MPP identified for a limit state function at the required reliability level, the R-percentile performance is given by

$$g^R = g(\mathbf{d}, \mathbf{u}_{MPP}) = g(\mathbf{d}, \mathbf{x}_{MPP}, \mathbf{p}_{MPP}). \tag{4}$$

Now the probabilistic optimization model in Eq. (1) becomes

$$\begin{aligned} &\textit{Find} \;\; \mathbf{d}, \boldsymbol{\mu}_x \\ &\textit{Minimize} \;\; f(\mathbf{d}, \boldsymbol{\mu}_x, \boldsymbol{\mu}_p) \\ &\textit{Subject to} \;\; g_i(\mathbf{d}, \mathbf{x}_{MPP}, \mathbf{p}_{MPP}) \geq 0 . \end{aligned} \tag{5}$$

The MPP in Eq. (5) is located by the inverse MPP search. Choi and Youn [33] developed different search algorithms for convex and concave functions. The Modified Advanced Mean Value (MAMV) method [35] was developed by us to improve the efficiency and robustness of the inverse MPP search for either convex or concave functions.

The employment of the equivalent deterministic optimization formulation allows us to use an efficient single loop strategy. With this strategy, deterministic optimization and reliability assessment are conducted in sequential series. The flowchart of SORA method is shown in Fig. 3. In each cycle, deterministic optimization is performed first, followed by reliability checking of $g_i(\mathbf{d}, \mathbf{x}_{MPP}, \mathbf{p}_{MPP}) \geq 0$. If not all probabilistic constraints satisfy the reliability requirements, the MPP information identified in the current cycle will be used to formulate the deterministic optimization in the next cycle. This procedure is repeated until convergence. The number of reliability assessment will be significantly reduced as it is now equal to the number of cycles used.

The deterministic optimization in the new cycle is formulated to improve the feasibility of a violated probabilistic constraint. This concept is illustrated in Fig. 4. As shown in the figure, if at the result of the deterministic optimization, the reliability constraint is not feasible, then its boundary $g_i(\mathbf{d}, \mathbf{x}_{MPP}, \mathbf{p}_{MPP}) = 0$ in the optimization model is shifted towards the feasible region by a small distance **s** based on the MPP recently found.

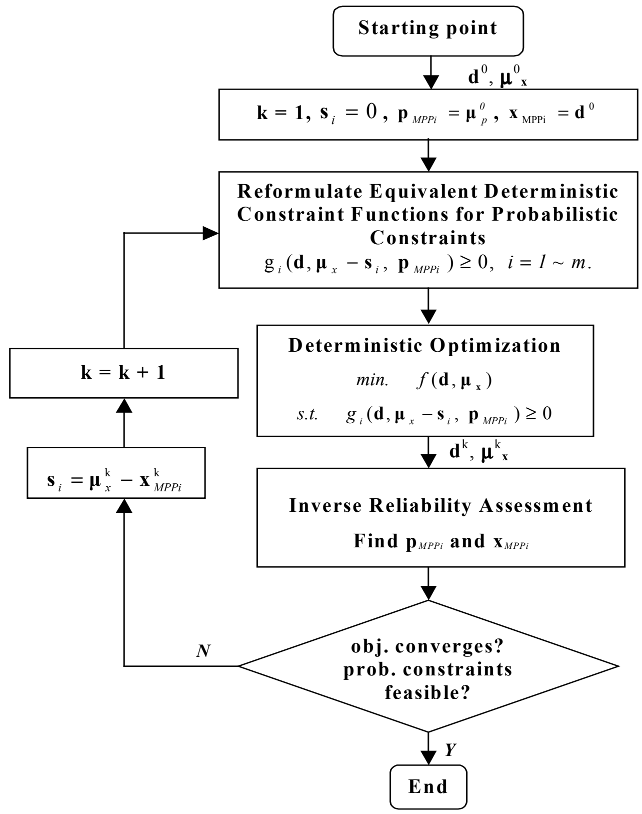

Figure 3. Flowchart of the SORA Method

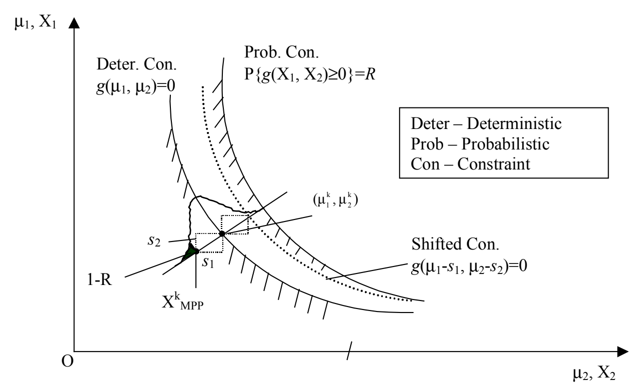

Figure 4. Shifting Distance of Probabilistic Boundaries [4]

In Fig. 4, the shifting distance (shown in dotted line) is applied so that the MPP is moved onto the deterministic boundary $g = 0$. By this way the feasible region of the next deterministic optimization is reduced. As the result, the reliability gets improved. The shifting distance is given by

$$\mathbf{s}_i^{k+1} = \boldsymbol{\mu}_x^{\,k} - \mathbf{x}_{i,\,MPP}^{k} . \tag{6}$$

The constraints in the deterministic optimization model are reformulated as

$$g_i\left(\mathbf{d}, \boldsymbol{\mu}_x - \mathbf{s}_i^{k+1}, \mathbf{p}_{i,\,MPP}\right) \geq 0, \quad i = 1, 2, ..., m \ . \tag{7}$$

The SORA method has been successfully applied to the reliability-based design for vehicle crashworthiness of side impact, and the integrated reliability and robust design of the speed reducer of a small aircraft engine [4,35]. Compared with existing double-loop methods, it is much more efficient with satisfactory accuracy. One of our current research goals is to further test the applicability of SORA method by applying it to structural design problems.

3. EXPLOITATION OF FEM ANALYSIS

In structural design problems, the displacement and stress fields are often used in evaluating the performance of the system. When uncertainties are involved in the system parameters, the first order derivatives of these functionals must be employed in solving the MPP problem. The finite differencing method for first order derivative is very computationally expensive and sometimes produces inaccurate solutions due to numerical errors. To overcome this problem, we will derive the first order derivatives of displacement and stress field analytically in this section.

Consider a general elastic body under applied load as shown in Fig. 5, the nodal displacement component along a direction **n** at position P_0 can be expressed as

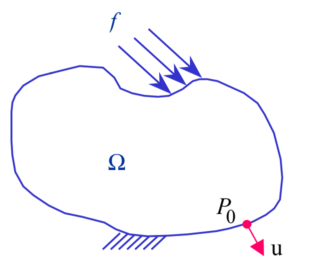

Figure 5. A General Elastic Body

$$u_{P_o} = \int_\Omega \delta(P - P_o)\mathbf{u} \cdot \mathbf{n} d\Omega \ , \tag{8}$$

where **u** is the displacement field and $\delta(P\text{-}P_0)$ denotes the Dirac-delta function, whose value is equal to 1 when point P coincides with P_0, otherwise is 0. The first derivative of the displacement component with respect to random material variables, e_i, can be represented as [45,46],

$$\frac{\partial u_{P_o}}{\partial e_i} = -\frac{1}{e_i}\mathbf{u}_i^T \mathbf{K}_i \mathbf{v}_i = -\frac{2}{e_i}E_i^m \ , \tag{9}$$

where $\mathbf{v}_i$ is the displacement under a unit load along direction **n** at position P_0 and E^m_i denotes the mutual energy of the two loading cases. The element mutual energy can also be expressed in terms of element strain energies as,

$$\begin{aligned} E_i^m &= \frac{1}{2}\left[\frac{1}{2}(\mathbf{u}_i + \mathbf{v}_i)^T \mathbf{K}_i (\mathbf{u}_i + \mathbf{v}_i) - \frac{1}{2}\mathbf{u}_i^T \mathbf{K}_i \mathbf{u}_i - \frac{1}{2}\mathbf{v}_i^T \mathbf{K}_i \mathbf{v}_i\right], \\ &= \frac{1}{2}\left(E_i^t - E_i - E_i^u\right) \end{aligned} \tag{10}$$

where, E^t_i, E_i *and* E^u_i represent the element strain energies under combined and individual load cases. In many structural applications, the location of P_0 is the same as the applied load. Therefore, the element mutual energy can be reduced to the element strain energy, E_i directly.

Displacement along a given direction is a simple scalar function whereas the stress field is a tensor. To convert the stress tensor into a scalar form, an average stress functional can be defined in an integral form as

$$G = \int_\Omega g(\sigma, e_i) d\Omega \ , \tag{11}$$

where Ω denotes the region where the stress functional is of interest and e_i is the random material variable. The first derivative of the stress functional can be derived by the adjoint method as,

$$\frac{\partial G}{\partial e_i} = \int_\Omega \frac{\partial g}{\partial e_i} d\Omega - \int_\Omega \varepsilon^p \sigma^a d\Omega \ , \tag{12}$$

where ε^p is pseudo initial strain field can be obtained from the initial strain field. σ^a is the adjoint stress field and the adjoint system is a pre-strained system by $dg/d\sigma$ with no external loadings. In the next section, reliability of two truss structures will be studied using these analytical derivatives.

4. TRUSS DESIGN PROBLEMS

Two truss design problems are selected to verify the effectiveness of our proposed methods for efficient probabilistic structural design. We select a 5-bar truss design and a 10-bar truss design to illustrate the benefits of savings when the problem dimension goes up. For each problem, four probabilistic design formulations are applied to illustrate the benefits of using the SORA method and the analytical derivatives calculation. These four formulations include (1) Double-Loop Strategy with Probabilistic Formulation (DLM_Pro), (2) Double-Loop Strategy with R-Percentile Formulation (DLM_Per), (3) SORA Method using finite-difference for derivative calculation (SORA_Num), and (4) SORA Method using analytical derivatives (SORA_Ana). Formulations (1) and (2) are implemented in both truss design problems using the standard double loop procedure and our MPP search algorithm.

4.1 EXAMPLE OF 5-BAR TRUSS DESIGN

A 5-bar truss is shown in Fig. 6. The structure is symmetric about node 3. A 1000 lb external load is applied at node 3. Node 1 is fixed on the ground with a revolute joint. There is a translation joint at node 4. Each bar has square cross section. The design objective is to reduce the cost of the structure, i.e., to reduce its volume. The side lengths of the cross-sectional area of five bars are random design variables because of the variations in manufacturing. Their mean values $\boldsymbol{\mu}_x$ are design variables. Young's modulus E of each bar is a random parameter because of the uncertainty due to material processing. All random variables are assumed to follow normal distributions. The lengths of five bars $\mathbf{L}$, and the external force acting at node 3 are constants. There are two categories of constraints in this problem. The normal stresses in the five bars

σ_{actual} should be no larger than the allowable values $\sigma_{allowable}$ determined by their material. The displacement at node 3 in Y-(vertical) direction, $\delta_{3,Y,actual}$, should be no larger than the allowable value $\delta_{3,Y,allowable}$. All these constraints are probabilistic constraints, which must meet the required reliability levels; 99.9% and 99.99% are tested in this work. In summary, the formulation of a reliability-based design is stated as:

Find μ_{x_i}, $i = 1,2,3,4,5.$

Minimize truss volume $= \sum_{i=1}^{5} \mu_{x_i}^2 L_i$

Subject to $P\{g(j) = \sigma_{j,allowable} - \sigma_{j,actual} \ge 0\} \ge R_j, \quad j = 1,\ldots,5.$

$P\{g(6) = \delta_{3,Y,allowable} - \delta_{3,Y,actual} \ge 0\} \ge R_6.$

Bounds of design varibles

$1.08\text{ in} \le \mu_{x_1}, \mu_{x_5} \le 1.32\text{ in}$

$0.72\text{ in} \le \mu_{x_2}, \mu_{x_3}, \mu_{x_4} \le 0.88\text{ in}$

Probabilistic distribution information

$\sigma_{x_i} = 0.88\text{ in}, \; i = 1 \sim 5.$

$\mu_{E_1} = \mu_{E_2} = \mu_{E_4} = \mu_{E_5} = 29\text{e+}06\text{ psi},$

$\mu_{E_3} = 58\text{e+}06\text{ psi},$

$\sigma_{E_1} = \sigma_{E_2} = \sigma_{E_3} = \sigma_{E_4} = \sigma_{E_5} = 1.0\text{e+}05\text{ psi},$

Deterministic Design Parameter

$L_1 = L_5 = 11.55\text{ ft}, \; L_2 = L_4 = 5.775\text{ ft}, \; L_3 = 1.00026\text{ ft}$

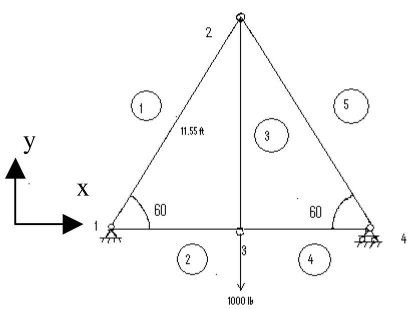

Figure 6. 5-bar Truss

For the 5-bar truss problem, there are in total six probabilistic constraints and 10 random variables. The stress sensitivity can be obtained from Eq. (12) by dropping out the first term. Also, since the displacement constraint is located at node 3 which is the same as the applied load, the sensitivity of this displacement function can be calculated easily with the element strain energy using the Eq. (9).

1) Case One: Desired Reliability = 99.9%

The cycle history of the probabilistic optimization using the No. (3) formulation, i.e., SORA_Num, is provided in Fig. 7. The SORA method only takes two probabilistic design cycles to converge in this case. In each cycle, the deterministic optimization, as shown in the solid line, is conducted first and then followed by the reliability assessment, as shown in the dashed line. We can see that in cycle one, the volume is optimized to its minimum value (457.4 in^3). However, the reliability assessment shows that the worst reliability of the probabilistic constraints (constraint 6 for displacement at node 3) is 99.7547%, which is not satisfactory (lower than 99.9%). In cycle two, the deterministic optimization is reformulated by applying the shifting distance (see Section 2.4). The optimal value of the volume changes to 461.3 in^3. This is slightly worse than that in cycle one; nevertheless, the worst reliability of probabilistic constraints now increases to 99.8999%, which is considered as feasible. As we can see from this example that the SORA method employs the strategy of gradually moving an infeasible probabilistic constraint to its feasible region. Due to the working principle of SORA, we can be sure that the critical probabilistic constraint(s) will always be at the boundary instead of being over-satisfied at the final cycle. It is noted in Fig. 7 that most of the function evaluations are used for reliability assessments (inverse MPP search). Among 225 total number of function evaluations, 187 are used for reliability assessments, while only 38 for deterministic optimization. One function evaluation stands for calling the analysis program once to get all the performance values related to the objective and constraints.

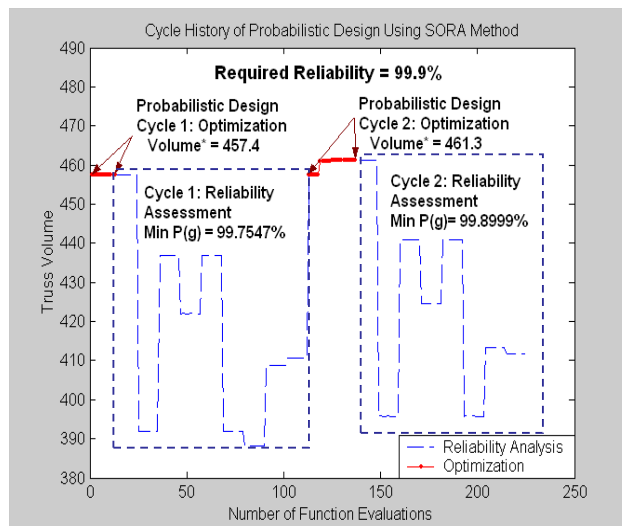

Figure 7. Cycle History of the Probabilistic Design Using SORA_Num (Case 1 — 5-bar Truss)

All four probabilistic design formulations are applied and their results are compared. It is found that all formulations lead to exactly the same design solution with the minimum mean volume as 461.3 inch^3 for this problem. The optimum values of the side lengths of bar 1 and 5 are both 1.08 inch. The optimum values of the side lengths of bar 2 and 4 are both 0.72 inch. The side length of bar 3 is 0.74232 inch. All design variables are located at their lower bounds except x_3. The displacement constraint for node 3 is active with reliability as 99.8999%. The results found by SORA are as accurate as those found by the double-loop methods but with much higher efficiency. A breakdown of the function evaluations for each formulation is provided in Table 1. In Table 1, it is noted that the DLM_Pro (Double Loop with Probabilistic Formulation) is the most computationally consuming method (total NFE = 8431) while the SORA method with analytical formulation for derivative calculation (SORA_Ana) is the most efficient one (total NFE =

71). Methods using the double-loop strategy, DLM_Pro and DLM_Per require much more function evaluations than the single-loop method, SORA. Percentile formulation is more efficient than the probabilistic formulation.

Table 1. Comparison of Number of Function Evaluations in Case 1 (5-Bar Truss)

Method	*NFE for Deterministic Optimization*	*NFE for Reliability Assessment*		*Total NFE*
		Derivative	*Other*	
DLM_Pro	–	–		8431
DLM_Per	–	–		2074
SORA_Num	38	187		225
		170	17	
SORA_Ana	38	33		71

NFE – Number of Function Evaluations.

If we take a close look at the breakdowns of the SORA_Num and the SORA_Ana methods in Table 1, we find that the NFE for deterministic optimization is the same. This is because both methods follow the same history of convergence and no analytical derivatives are used for deterministic optimization (for the ease of integrating with the existing optimization solver). On the other hand, the reliability assessments benefit greatly from using analytical derivatives. The NFE used for derivative calculations with the SORA_Num method is slightly over 10 times of that with the SORA_Ana method, because the starting point in MPP search is slightly different in two formulations due to the difference in numerical precision. The saving is approximately linear proportional to the number of random variables in this case.

2) Case Two: Desired Reliability = 99.99%

In this case, a higher desired reliability is assigned. We found that all four probabilistic formulations obtain the same solution with the optimum mean volume as 470.658 inch^3. The optimum values of side lengths of bar 1 and 5 are both 1.08 inch. The optimum values of side lengths of bar 2 and 4 are both 0.72 inch. The optimum value of side length of bar 3 is 0.7930 inch. The displacement constraint at node 3 is again the only active constraint with reliability as 99.99%. If we verify the actual reliabilities of all other probabilistic constraints, we find that all stress constraints have the reliability of 1.0. That explains why the method based on probabilistic formulation requires such a large number of function evaluations. Compared with Case 1, it is noted that objective value gets worse while the reliability of the design increases from 99.9% to 99.99%. In Case 2, three cycles are conducted if using the SORA method.

A breakdown of the function evaluations for each formulation is provided in Table 2. Similar observations with Case 1 are made regarding the relative performance of each formulation. In general, more computations are needed for each formulation compared to Case 1 because higher desired reliability is set. For this case, the savings by using analytical derivatives is also slightly over 10 times (240 vs. 22), which indicates that the saving is approximately linear proportional to the number of random variables.

Table 2. Comparison of Number of Function Evaluations in Case 2 (5-Bar Truss)

Method	*NFE for Deterministic Optimization*	*NFE for Reliability Assessment*		*Total NFE*
		Derivative	*Other*	
DLM_Pro	–	–		9513
DLM_Per	–	–		2553
SORA_Num	57	264		321
		240	24	
SORA_Ana	57	46		103
		22	24	

4.2 EXAMPLE OF A 10-BAR TRUSS DESIGN

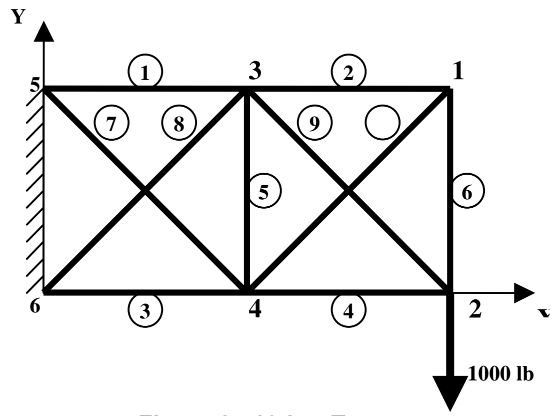

Figure 8. 10-bar Truss

A 10-bar truss is shown in Fig. 8. Nodes 5 and 6 are fixed on the ground. The lengths of bars 1, 2, 3, 4, 5, and 6 are equal, while those of bars 7, 8, 9 10 are equal. The bar lengths L_i are constants. A 1000 lb loading is applied at node 2 along the negative Y direction. The side lengths of all bars are random design variables. The Young's modulus of each bar is a random parameter. All random variables are assumed to follow normal distributions. All constraints, i.e., stress in each bar and displacement at node 2 along the negative Y (vertical) direction, are probabilistic constraints, with required reliability as 90%. The probabilistic design model is described as follows:

Find μ_{x_i}, $i = 1,2,...,10.$

Minimize truss volume $= \sum_{i=1}^{10} \mu_{x_i}^2 L_i$

Subject to $P\{g(j) = \sigma_{j,\,allowable} - \sigma_{j,actual} \geq 0\} \geq R_j$, $j = 1,2,...,10.$

$P\{g(11) = \delta_{2,Y,allowable} - \delta_{2,Y,actual} \geq 0\} \geq R_{11}.$

Bounds of design varibles

$1.2\,\text{in} \leq \mu_{x1},...,\mu_{x10} \leq 4.0\,\text{in}$

Probabilistic distribution information

$\sigma_{x_i} = 0.2\,\text{in}, \quad i = 1,...,10.$

$\mu_{E_i} = 3\text{e+}04\,\text{psi}, \quad \sigma_{E_i} = 0.5\text{e+}03\,\text{psi}, \quad i = 1,...,10.$

Deterministic Design Parameter

$L_1 = ... = L_6 = 30\,\text{ft}, \; L_7 = ... = L_{10} = 42.4264\,\text{ft}$

Four probabilistic design formulations are again applied. It is observed that the DLM_Pro, SORA_Num and the

SORA_Ana methods generate very similar results with the minimum mean truss volume as 12658.38 inch^3. The only active probabilistic constraint is constraint 11, the displacement at node 2. Because there are 20 probabilistic constraints and 20 random variables in this problem, the computation demands are very large for the DLM_Pro method (NFE=1,076,174). When running the DLM_Per formulation, optimization does not converge because the search is trapped in a local but infeasible point. This is because the DLM_Per formulation poses more challenges on the MPP search. When multiple local optimal MPPs exist, an ineffective MPP search will lead the probabilistic optimization to a wrong direction.

It is noted in Table 3 that the SORA method with the analytical derivatives (SORA_Ana) is much more efficient than the other formulations. Compared to the 5-bar truss problem, the benefits of using the SORA method for the 10-bar truss design are more significant because there are more (inactive) probabilistic constraints involved.

Table 3. Comparison of Number of Function Evaluations (10-Bar Truss)

<table>
<tr><th rowspan="2">Method</th><th rowspan="2">NFE for Deterministic Optimization</th><th colspan="2">NFE for Reliability Assessment</th><th>Total NFE</th></tr>
<tr><th>Derivative</th><th>Other</th><th></th></tr>
<tr><td>DLM_Pro</td><td>–</td><td colspan="2">–</td><td>1,076,174</td></tr>
<tr><td>DLM_Per</td><td colspan="4">Optimization does not converge</td></tr>
<tr><td rowspan="2">SORA_Num</td><td rowspan="2">836</td><td colspan="2">1262</td><td rowspan="2">2098</td></tr>
<tr><td>1200</td><td>62</td></tr>
<tr><td rowspan="2">SORA_Ana</td><td rowspan="2">860</td><td colspan="2">76</td><td rowspan="2">936</td></tr>
<tr><td>37</td><td>39</td></tr>
</table>

The number of function evaluations for finite differencing derivative calculations will increase with the number of probabilistic constraints and the problem dimension. As shown in Table 3, the benefits of using the analytical derivative calculation again shown to be significant when there are large number of probabilistic constraints and random variables in the probabilistic design problem. For this case, even though there are 20 random variables, the savings by using analytical derivatives is much more over 20 times (1200 vs. 37). We find the reason is because the history of the deterministic optimization and the MPP search for reliability assessment are quite different between the SORA_Num and the SORA-Ana methods due to the difference in numerical precision. The SORA_Ana is shown to be much more efficient for convergence in the MPP search process.

5. CONCLUSIONS

The computational costs associated with the reliability analysis and performance evaluations using the finite element method have restricted the applications of probabilistic optimization in structural design problems. By examining the major factors contributing to the computations, we develop methods that take the corresponding measures to significantly improve the efficiency of solving probabilistic structural design problems. In particular, the SORA method is applied as a new probabilistic optimization strategy to significantly reduce the amount of reliability assessments required; a finite element based analytical derivative formulation is developed and integrated with SORA to further reduce the computation effort.

Our testing examples illustrate that the single loop strategy offered by the SORA method can avoid the nested optimization synthesis and reliability assessment loops that exist in the conventional probabilistic optimization techniques. The SORA method works effectively to quickly bring the solution that violates the probabilistic constraints to the feasible region. The replacement of the probability evaluation by the R-percentile formulation is especially useful for improving the efficiency when there are a large number of probabilistic constraints but majority are inactive. By using analytical formulations for derivative calculations, instead of using the finite differencing approach, the accuracy of derivative evaluations is increased and the efficiency of probabilistic optimization is further improved because of ease of convergence. The minimum savings of using analytical derivatives is proportional to the number of random variables in a problem. In large-scale structural design problems, the savings is expected to be more because the probabilistic formulations with analytical derivatives converge faster in MPP search due to the improved numerical precision. The computational cost can be further reduced if analytical derivatives are employed for the deterministic optimization in each probabilistic optimization cycle.

ACKNOWLEDGMENTS

The grants from National Science Foundation NSF 0099775 (UIC) and NSF 0100024 (Rutgers University) are gratefully acknowledged.

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