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Enhanced Sequential Optimization and Reliability Assessment Method for Changing Design Variance

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ABSTRACT

The Sequential Optimization and Reliability Assessment (SORA) method is a single loop method containing a sequence of cycles of decoupled deterministic optimization and reliability assessment for improving the efficiency of probabilistic optimization. However, the original SORA method as well as some other existing single loop methods is not efficient for solving problems with changing variance. In this paper, to enhance the SORA method, three formulations are proposed by taking the effect of changing variance into account. These formulations are distinguished by the different strategies of Inverse Most Probable Point (IMPP) approximation. Mathematical examples and a pressure vessel design problem are used to test and compare the effectiveness of the proposed formulations. The "Direct Linear Estimation Formulation" is shown to be the most effective and efficient approach for dealing with problems with changing variance. The gained insight can be extended and utilized to other optimization strategies that require MPP or IMPP estimations.

KEYWORDS

probabilistic optimization, single loop method, most probable point, changing variance.

NOMENCLATURE

CDF = Cumulative Distribution Function
DLP = Double Loop
DO = Deterministic Optimization
FORM = First Order Reliability Method
IMPP = Inverse Most Probable Point
KKT = Karush-Kuhn-Tucker
MAMV = Modified Advanced Mean Value
MPP = Most Probable Point
PDF = Probability Density Function
RA = Reliability Assessment
SLP = Single Loop
SORA = Sequential Optimization and Reliability Assessment
SORM = Second Order Reliability Method
$\boldsymbol{d}$ = deterministic design vector
g_i = i^{th} probabilistic constraint
$\boldsymbol{p}$ = random parameter vector
R_i = reliability level for the i^{th} probabilistic constraint
$\boldsymbol{s}$ = shifting vector
$\boldsymbol{u}$ = independent and standardized normal vector
U_i = i^{th} component of $\boldsymbol{u}$
$\boldsymbol{x}$ = random design vector
X_i = i^{th} random design variable
$\boldsymbol{x_{IMPP}}$ = IMPP vector
X_{IMPPi} = i^{th} component of IMPP vector
$\boldsymbol{u_{IMPP}}$ = IMPP vector in standard normal space
U_{IMPPi} = i^{th} component of IMPP in standard normal space
β = reliability index
$\boldsymbol{\mu_x}$ = vector of mean values of random design variables
μ_{x_i} = mean of i^{th} random design variable
σ_{x_i} = standard deviation of i^{th} random design variable
r_i = coefficient of variation for i^{th} random design variable

1. INTRODUCTION

Probabilistic design optimization offers an approach for making reliable design decisions with the consideration of uncertainty [1-7]. In the existing applications, probabilistic optimization is accomplished by either the Double Loop (DLP) method [8-11] or the Single Loop (SLP) method [12-15]. The traditional double loop method consists of the design optimization loop (outer loop) in the original design variable space, which iteratively performs the reliability analysis (inner loop) in the standard normalized space of random design variables until achieving the optimum and meeting the desired reliability of probabilistic constraints. Due to the nested structure of outer and inner loops, the double loop method is often computationally expensive and not affordable for a variety of practical applications.

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Recent years have seen many efforts made towards developing new and efficient probabilistic optimization strategies. The single loop method has been developed to avoid the nested coupling of the outer and inner loops [12, 14, 15]. The existing SLP methods deviate in the way of how optimization and reliability analysis are organized. One type of SLP methods directly integrate the reliability analysis into a design optimization procedure and solve the integrated problem as a single optimization problem. The other type of SLP methods decouple the optimization and reliability analysis and solve them sequentially from cycle to cycle. We call the former "integrated single loop method" and the later "decoupled, sequential single loop method". An example under the "integrated single loop method" is the "single loop single vector" approach [14] that is employed in a reduced, uncorrelated and normalized parameter space other than the standard normal space. Without conducting the expensive Most Probable Point (MPP) [16] search, it uses the steepest descent direction from the previous iteration to approximate MPP and generates an equivalent deterministic optimization problem. Although this method avoids nested loops, it needs to evaluate all the constraint derivatives to identify the active constraints by using an active constraint searching strategy. Another efficient method under the "integrated single loop method" is the approach recently developed by Liang et al. [15], where a probabilistic optimization problem is converted into an equivalent deterministic optimization problem by adding an equality constraint that determines the MPP based on the Karush-Kuhn-Tucker (KKT) optimality conditions of the reliability constraints. Although the method greatly improves the efficiency by eliminating the reliability analysis loop, the KKT condition is only a necessary but not sufficient condition. The method may have difficulties in locating the exact MPP when there are multiple, local MPPs.

Examples of the "decoupled, sequential single loop methods" include the Safety-Factor Approach [17] and our recently developed Sequential Optimization and Reliability Assessment (SORA) method [12]. The SORA method [12] decouples the probabilistic optimization problem in the form of a serial of sequential Deterministic Optimization (DO) followed by Reliability Assessments (RA). The SORA method achieves high efficiency by taking the following measures: the use of R-percentile formulation (Inverse Most Probable Point - IMPP concept) to evaluate constraint satisfaction only up to the required reliability level (R) [18]; the use of an efficient and robust IMPP search algorithm for reliability assessment [19]; and the employment of a shifting vector for estimating IMPP in the DO formulation of a new cycle. Because the IMPP estimation of a new cycle is always based on the *exact* IMPP from the previous cycle, the method improves the accuracy in IMPP estimation, which is especially useful when there are multiple, local optimal solutions to IMPPs. The safety-factor approach [17] follows the similar strategy of decoupling optimization and reliability assessment, but approximates the MPP by performing a reliability analysis with the "shifted" limit-sate function. Yang and Gu [20] compared various probabilistic optimization strategies on a set of benchmarking problems. It shows that the SORA method is one of the most robust and efficient methods.

One limitation of the SORA method and some other existing probabilistic optimization methods is that the existing formulation and solution strategies only take into account the random design variables with constant variances. In practice, instead of fixed variance, design variance often depends on the magnitude of a design variable. For example, the concentricity of the cylindric surface of a shaft needs to be controlled within a certain range if the geometry of the shaft is that of a stepped cylinder. Compared with a small shaft, to manufacture one with larger size in proportion, larger variance of the concentricity is allowed due to the increasing difficulties in keeping the same concentricity when the length of the shaft becomes larger. One efficient way to describe the changing variance is to introduce the *coefficient of variation* (the rate of standard deviation to the mean value of a random variable). The changing variance in probabilistic design poses more difficulties in predicting the MPP and the constraint boundary of an equivalent DO formulation as well as the final design solution.

Our objective in this paper is to enhance the SORA method by developing new formulations that take into account the changing variance, while keeping the decoupling, single loop strategy the same. Three different formulations are examined in this work, and compared to the use of the original SORA method for problems with changing variance. Observations and recommendations are made based on comparative studies using several illustrative examples.

The paper is organized as follows. Section 2 of this paper provides a brief review of the original SORA method as well as the concept of Inverse Most Probable Point (IMPP). In Section 3, four formulations that employ different strategies to consider the effect of changing variance are presented. They are 1) Shifting Vector Formulation (Approach 0 or the original SORA method); 2) Approximated $\boldsymbol{u}_{IMPP}$ Formulation (Approach 1); 3) Direct Linear Estimation Formulation (Approach 2); and 4) Quasi First Order Approximation of $\boldsymbol{x}_{IMPP}$ Formulation (Approach 3). The efficiencies and optimal results of these methods are compared in Section 4 through illustrative examples. Section 5 provides the conclusion.

2. REVIEW OF THE SORA METHOD

2.1 BACKGROUND OF PROBABILISTIC OPTIMIZATION

A typical probabilistic design model [21] is defined as

$$\begin{aligned}&\text{Find} \quad \boldsymbol{d}, \boldsymbol{\mu}_x\\&\text{Minimize} \quad f(\boldsymbol{d}, \boldsymbol{x}, \boldsymbol{p})\\&\text{Subject to} \quad P_i\{g_i(\boldsymbol{d}, \boldsymbol{x}, \boldsymbol{p}) \geq 0\} \geq R_i, \ \mathrm{i}=1,2,\cdots,\mathrm{m}\end{aligned} \quad , \quad (1)$$

where $\boldsymbol{d}$ vector stands for deterministic design variables; $\boldsymbol{x}$ vector contains all random design variables X_i; $\boldsymbol{p}$ vector is for random design parameters with fixed distributions; m is the number of reliability constraints; and $\boldsymbol{\mu}_{\boldsymbol{x}}$ is a vector composed of mean values of random design variables. Different from deterministic design, probabilistic design focuses on maintaining design feasibility for constraints at desired probabilistic levels. Thus the probability of constraint satisfaction, P_i for $g_i \geq 0$, should be no less than the desired reliability level R_i, which is referred to reliability assessment, The g_i functions are the so-called limit state functions [16].

One efficient way for reliability assessment is to employ the Most Probable Point (MPP) approach [16]. With the MPP approach, all random variables ***x*** and ***p*** are transformed into ***u*** in a standardized normal space, called ***u***-space. In ***u***-space, MPP is formally defined as a point located on the limit state function hypersurface with the minimum distance to the origin point. The corresponding minimum distance is the safety index β indicating the reliability level of the limit state function as shown in Fig. 1. When the First Order Reliability Method (FORM) is used, reliability R is approximated as the standardized normal Cumulative Distribution Function (CDF) $\Phi(\beta)$. Higher order estimations, such as Second Order Reliability Method (SORM) [22-24], could be used if a more precise estimation of reliability is needed.

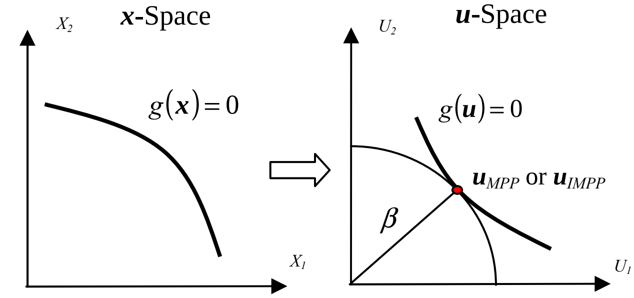

Figure 1. Transformation of input variables and illustration of most probable point (MPP)

Determining the MPP point on the limit state function, or called the original MPP search, can be implemented by solving the following optimization problem in Eqn. (2).

$$\min_{U} \|\boldsymbol{u}\| \quad s.t.\ \ g(\boldsymbol{u}) = 0 \quad , \qquad (2)$$

where $\|\boldsymbol{u}\|_{\min}$ is the safety index β. Inversely, given a desired reliability level R and correspondingly the safety index β, the procedure of finding the corresponding MPP on the limit state function is called the Inverse MPP (IMPP) search. Among the IMPP search methods, the Modified Advanced Mean Value (MAMV) method [25] improves the efficiency and robustness of the IMPP search for both convex and concave limit state functions. The IMPP facilitates the conversion of original reliability constraints to R-percentile formulations for improving computational efficiency. At the MPP as well as IMPP, for a normal distribution function, the following relationship holds when transforming random variables ***x*** into the normal space ***u***:

$$X_{IMPPi} = \mu_{x_i} + \sigma_{x_i} \cdot U_{IMPPi} \quad . \qquad (3)$$

2.2 THE SORA METHOD

The Sequential Optimization and Reliability Assessment (SORA) method is a decoupled, sequential single loop method for probabilistic optimization [12]. As shown in Fig. 2, SORA improves the computational efficiency of probabilistic optimization by decoupling the reliability assessment from the optimization loop. In each cycle k, the procedure contains two separate parts of Deterministic Optimization (DO) and Reliability Assessment (RA). The DO at cycle k is formulated by including the predicted IMPP which is estimated based on the exact IMPP verified from the RA in cycle k-1, marked as a dash line in the flowchart. In each cycle, the design solution obtained from DO is verified by checking the feasibility of probabilistic constraint in RA. If the feasibility is satisfied, the process will stop after verifying the convergence criterion; otherwise the cycle will be repeated.

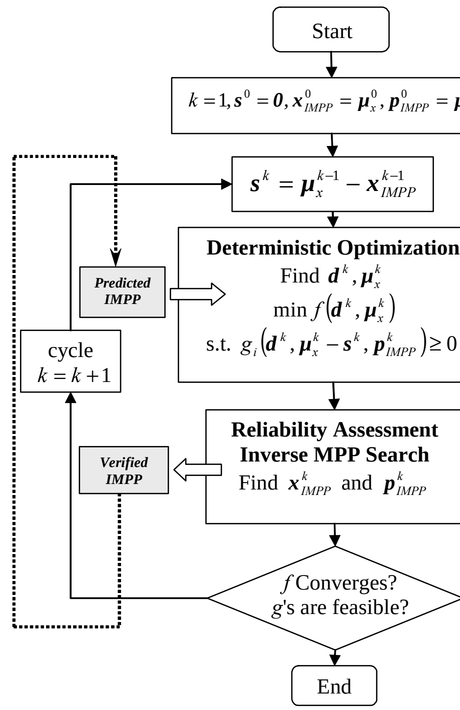

Figure 2. Flowchart of the original SORA method

Two important measures are taken in converting probabilistic constraints to equivalent deterministic constraints in the DO formulation of SORA: one is related to the use of R percentile formulation to replace the original reliability constraint; the other applies a shifting vector to refine the feasible region if the design solution from DO is verified to be infeasible from RA. Given the desired reliability R, the original expression of a constraint g in probabilistic design is $P\{g(\boldsymbol{d}, \boldsymbol{x}, \boldsymbol{p}) \geq 0\} \geq R$. The percentile performance g^R, is the value of the limit state function g that makes the integration area under the probabilistic distribution function of g for $g \geq g^R$ exactly equal to the required reliability R. The original probabilistic constraint can be rewritten in the equivalent R-percentile formulation as

$$g^R \geq 0, \text{ and further } g^R = g(\boldsymbol{d}, \boldsymbol{x}_{IMPP}, \boldsymbol{p}_{IMPP}) \geq 0. \qquad (4)$$

The use of R-percentile formulation saves the computational effort by evaluating the design feasibility only up to the *desired* reliability level (R), which is often lower than the *actual* reliability level when a probabilistic constraint is feasible. It also converts the original probabilistic formulation to an equivalent deterministic optimization as:

$$\begin{array}{ll} \text{Find} & \boldsymbol{d}, \boldsymbol{\mu}_x \\ \text{Minimize} & f(\boldsymbol{d}, \boldsymbol{\mu}_x, \boldsymbol{p}) \\ \text{Subject to} & \boldsymbol{g}(\boldsymbol{d}, \boldsymbol{x}_{IMPP}, \boldsymbol{p}_{IMPP}) \geq 0 \end{array} \quad . \quad (5)$$

The model in Eqn.(5) shows that for a critical probabilistic constraint, the IMPPs should sit on the boundary of the deterministic constraint. Since the exact IMPPs of $\boldsymbol{x}$ and $\boldsymbol{p}$ at the design solution $(\boldsymbol{d}, \boldsymbol{\mu}_x)$ of the current DO are not known until the RA is implemented, they are approximated based on the exact IMPPs from the RA in the previous cycle. With the original SORA method, a shifting vector concept is applied and the i^{th} constraint of the DO formulation in Cycle k+1 becomes

$$g_i\left(\boldsymbol{d}, \boldsymbol{\mu}_x - \boldsymbol{s}^{k+1}, \boldsymbol{p}_{IMPP}^{k+1}\right) \geq 0 \quad , \quad (6)$$

where the shifting vector $\boldsymbol{s}^{k+1}$ in Cycle k+1 is defined as

$$\boldsymbol{s}^{k+1} = \boldsymbol{\mu}_x^k - \boldsymbol{x}_{IMPP}^k \;\text{ or }\; s_i^{k+1} = -\sigma_{x_i} \cdot U_{IMPPi}^k \quad . \quad (7)$$

The idea behind using the shifting vector is illustrated in Fig. 3 for a problem having two random design variables X_1, X_2 with means being μ_1 and μ_2, respectively. It shows that when the exact IMPP from RA in cycle k, i.e., $\boldsymbol{x}_{IMPP}^k$, falls to the left side of DO constraint, the probabilistic constraint is infeasible. To ensure that the IMPP is located on the deterministic constraint boundary, in the next cycle k+1, the new boundary marked as the dotted line is moved from the previous deterministic constraint boundary by the shifting vector $\boldsymbol{s}^{k+1}$. We can also interpret the shift concept by relating Eqns. (4) and (6) to the R-percentile formulation of constraints at cycle $k+1$, i.e.,

$$g_i\left(\boldsymbol{d}, \boldsymbol{x}_{IMPP}^{k+1}, \boldsymbol{p}_{IMPP}^{k+1}\right) \geq 0 \quad . \quad (8)$$

Relating to Eqns. (6) and (7), we are basically using the following predictions for the IMPP at Cycle k+1 with the SORA method:

$$\boldsymbol{x}_{IMPP}^{k+1} \approx \boldsymbol{\mu}_x^{k+1} - \boldsymbol{s}^{k+1} = \boldsymbol{\mu}_x^{k+1} - (\boldsymbol{\mu}_x^k - \boldsymbol{x}_{IMPP}^k) = \boldsymbol{\mu}_x^{k+1} - \boldsymbol{\mu}_x^k + \boldsymbol{x}_{IMPP}^k \;, (9)$$

$$\boldsymbol{p}_{IMPP}^{k+1} \approx \boldsymbol{p}_{IMPP}^k \quad . \quad (10)$$

As more mathematical details will be revealed later in Section 3.3, Eqns. (9) and (10) provide accurate estimations of IMPPs when the limit state function is linear in the $\boldsymbol{x}$ domain and when the variances of $\boldsymbol{x}$ are constants (variance of $\boldsymbol{p}$ is constant by default). Under these conditions, the SORA method should find the optimal probabilistic solution in cycle 2. Under general conditions, it often takes a few more cycles to reach the final solution while the feasibility of a solution progressively improves and the estimations of IMPPs get closer to the real values.

The SORA method has been successfully applied to design applications such as vehicle crashworthiness and other engineering problems [12, 26]. Compared to the double loop method, the SORA method is shown to be much more efficient. However, the original SORA method does not incorporate the information of the design point and design variance in a new cycle. For general conditions, especially in the case that the variance of a design variable is changing with respect to the mean location, improved formulations need to be developed to achieve higher efficiency for convergence.

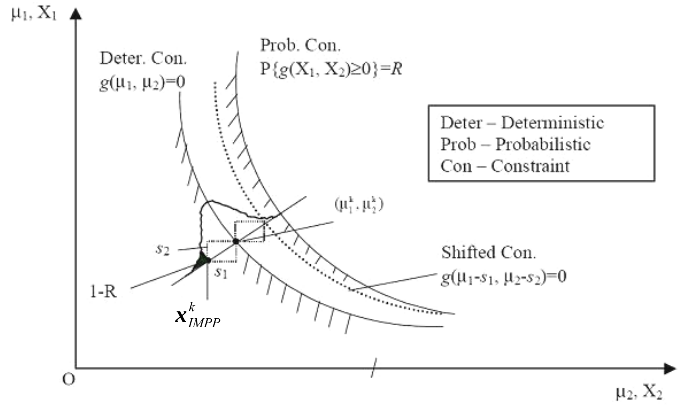

Figure 3. Shifting distance of probabilistic boundaries

3. ENHANCING SORA FOR PROBLEMS WITH CHANGING VARIANCES

In this work, we consider three new formulations to incorporate changing variances into the formulation of SORA method. The original shifting vector formulation in SORA, named as Approach 0, is also tested for its efficiency in dealing with changing variance. All new approaches (Approaches 1-3) follow the same strategy of decoupling the deterministic optimization (DO) and reliability assessment (RA) as in SORA. They deviate in the formulations for predicting the IMPP in a new cycle. Empirical results are provided in Section 4.

3.1 APPROACH 0 (ORIGINAL SORA): SHIFTING VECTOR FORMULATION

This approach uses the same DO formulation as in the original SORA method (Eqn. (6)), but updates the variance right before the RA when verifying the exact IMPP. Assuming that the deviation of a random design variable is a function of its mean value, the core part of Approach 0 is shown in Fig. 4. In principle, this approach should be able to progressively improve the design solution by shifting the constraint boundary to the feasible region based on the exact IMPP obtained from the most recent cycle. However, because the changing variance increases the computational effort for the IMPP search and the formulation used in DO does not incorporate the changing variance information, this approach is expected to take more cycles to reach the final optimal solution compared to solving problems with constant variance.

At Cycle k+1

1. Define shifting vector $\boldsymbol{s}^{k+1} = \boldsymbol{\mu}_x^k - \boldsymbol{x}_{IMPP}^k$
2. Deterministic Optimization to find $\boldsymbol{d}^{k+1}$ and $\boldsymbol{\mu}_x^{k+1}$

$$\min \; f\left(\boldsymbol{d}^{k+1}, \boldsymbol{\mu}_x^{k+1}\right)$$
$$\text{s.t. } g_i\left(\boldsymbol{d}^{k+1}, \boldsymbol{\mu}_x^{k+1} - \boldsymbol{s}^{k+1}, \boldsymbol{p}_{IMPP}^k\right) \leq 0$$

3. Update deviation as $\sigma_{x_i}^{k+1} = \sigma_{x_i}(\mu_{x_i}^{k+1})$
4. Reliability Assessment to find $\boldsymbol{x}_{IMPP}^{k+1}$ and $\boldsymbol{p}_{IMPP}^{k+1}$

Figure 4. Core of Cycle k+1 of Approach 0

3.2 APPROACH 1: APPROXIMATED u_{IMPP} FORMULATION

Based on Eqn. (3), the following relationship should hold at cycle k+1:

$$X_{IMPPi}^{k+1} = \mu_{x_i}^{k+1} + \sigma_{x_i}^{k+1} \cdot U_{IMPPi}^{k+1} \quad . \quad (11)$$

Approach 1 approximates the IMPP at a new cycle by replacing U_{IMPPi}^{k+1} in the above equation by U_{IMPPi}^{k}, i.e.,

$$X_{IMPPi}^{k+1} \approx \mu_{x_i}^{k+1} + \sigma_{x_i}(\mu_{x_i}^{k+1}) \cdot U_{IMPPi}^{k} \quad . \quad (12)$$

The flow chart of approach 1 is similar to that of approach 0 in Fig. 4, except that the shifting vector is now changed to

$$s_i^{k+1} = -\sigma_{x_i}(\mu_{x_i}^{k+1}) \cdot U_{IMPPi}^{k} \quad . \quad (13)$$

With approach 1, the variance information is incorporated by updating the standard deviation in Eqn. (13) as the function of the mean value, $\sigma_{x_i}(\mu_{x_i})$, in a new cycle. However, since the $\boldsymbol{U}_{IMPP}$ information is estimated based on the exact IMPP from the previous cycle, the formulation only provides an approximation. It can be expected that if $\boldsymbol{U}_{IMPP}^{k+1}$ is not too much far away from $\boldsymbol{U}_{IMPP}^{k}$, Approach 1 should work well and converge quickly.

3.3 APPROACH 2: DIRECT LINEAR ESTIMATION FORMULATION

The proposed Approach 2 derives the $\boldsymbol{x}_{IMPP}^{k+1}$ by considering the slope information of the linearized limit state function $g(\boldsymbol{x})$ at the exact IMPP from RA cycle k. Consider a linear constraint

$$g(\boldsymbol{x}) = \sum_{i=1}^{n} a_i X_i + a_0 \quad . \quad (14)$$

where a_i are the constraint function coefficients and $\boldsymbol{x} = (X_1, \cdots, X_n)$ is the design variable vector composed of n independent random variables with normal distributions. After transforming $\boldsymbol{x}$ to $\boldsymbol{u} = (U_1, \cdots, U_n)$ in $\boldsymbol{u}$-space using $U_i = \frac{X_i - \mu_{x_i}}{\sigma_{x_i}}$, the limit state function in $\boldsymbol{u}$- space becomes

$$\begin{aligned} g(\boldsymbol{U}) &= \sum_{i=1}^{n} a_i (\sigma_{x_i} U_i + \mu_{x_i}) + a_0 \\ &= \sum_{i=1}^{n} b_i U_i + b_0 \end{aligned} \quad . \quad (15)$$

where $b_i = a_i \sigma_{x_i}$ and $b_0 = \sum_{i=1}^{n} a_i \mu_{x_i} + a_0$, $\boldsymbol{b} = [b_1, \cdots, b_n]$. As shown in Fig. 1, the IMPP is the point where the norm of $\boldsymbol{u}$ reaches its minimum of

$$\beta = \|\boldsymbol{u}\|_{\min} = \|\boldsymbol{u}^*\| = \frac{|b_0|}{\|\boldsymbol{b}\|} \text{ when } \boldsymbol{u} = \boldsymbol{u}^* \quad . \quad (16)$$

Here β satisfies the equation of $\beta = \Phi^{-1}(R)$; R is the desired reliability for the constraint and $\|\cdot\|$ is the norm of an n-dimensional vector; and $\boldsymbol{u}^*$ is a vector starting from the origin and perpendicular to the hypersurface with the norm β and the direction $\hat{\boldsymbol{b}} = \frac{-\boldsymbol{b}}{\|\boldsymbol{b}\|}$. The IMPP in the $\boldsymbol{u}$-space can be written as

$$U_{IMPPi} = U_i^* = \|\boldsymbol{u}^*\| \cdot \left(\frac{-\boldsymbol{b}}{\|\boldsymbol{b}\|}\right)_i = \frac{-b_i \beta}{\|\boldsymbol{b}\|} \quad . \quad (17)$$

Hence for any IMPP,

$$X_{IMPPi} = \mu_{x_i} + \sigma_{x_i} U_{IMPPi} = \mu_{x_i} - \frac{b_i \beta \sigma_{x_i}}{\|\boldsymbol{b}\|} \quad . \quad (18)$$

From Eqn. (18), we can see that for random design variables with *constant* variances, if the slope, i.e., $\boldsymbol{a} = [a_1, \cdots, a_n]$, of the linearized limit state functions doesn't change too much when the IMPP moves from cycle to cycle, the second item in Eqn. (18) is almost a fixed value no matter where $\boldsymbol{\mu_x}$ is. This explains the reason why in the original SORA method, for constant variance, the difference between $\boldsymbol{x}_{IMPP}$ and $\boldsymbol{\mu_x}$ is treated as a constant vector. When the variance is changing, we use Eqn. (18) to predict IMPP in a new cycle k+1 as,

$$X_{IMPPi}^{k+1} \approx \mu_{x_i}^{k+1} - \frac{b_i \beta \sigma_i(\mu_{x_i}^{k+1})}{\|\boldsymbol{b}\|} = \bar{h}_i\left(\boldsymbol{\mu}_x^{k+1}\right), \quad (19)$$

where $\boldsymbol{b}$ is obtained by linearizing the $g(\boldsymbol{x})$ function at the exact IMPP from RA in cycle k. The whole prediction function is represented as $\bar{h}_i(\boldsymbol{\mu}_x)$. The bar above h_i means the prediction is based on linearizing a limit state function.

Based on its principle, Approach 2 is expected to solve a probabilistic optimization problem with linear constraints and constant variance in no more than 3 cycles: the first cycle for the initial deterministic optimization, the second cycle for locating a design solution with IMPP exactly at the boundary of DO, and the last one for convergence check. For nonlinear constraints, it is expected that Approach 2 should be quite efficient if the slope of a limit state function does not change too much when the IMPP moves from cycle to cycle. Since the slope of $\boldsymbol{g}$ at the IMPP has been obtained in RA during the IMPP search of the last cycle, Approach 2 does not require additional function calls.

3.4 APPROACH 3: QUASI FIRST ORDER APPROXIMATION OF x_{IMPP} FORMULATION

Approach 1 utilizes the direct information of the exact IMPP from the previous cycle, and Approach 2 uses the slope information of a limit state function at the exact IMPP, to estimate the IMPP in a new cycle. We consider here Approach 3 that employs the Taylor expansion of the IMPP in $\boldsymbol{x}$-space and provides the estimation of IMPP based on both the information of the exact IMPP and the slope of a limit state function from the previous cycle.

For nonlinear functions, suppose $\boldsymbol{x}_{IMPP}$ is a function of $\boldsymbol{\mu_x}$ as $\boldsymbol{x}_{IMPP} = h(\boldsymbol{\mu_x})$, no matter whether the variance is fixed or changing. Here the function h is unknown and cannot be expressed explicitly. In DO formulation of cycle k+1, let

$$\begin{aligned} \boldsymbol{x}_{IMPP}^{k+1} &= h\left(\boldsymbol{\mu}_{\boldsymbol{x}}^{k+1}\right) \approx h\left(\boldsymbol{\mu}_{\boldsymbol{x}}^{k}\right) + \frac{\partial h^k}{\partial \boldsymbol{\mu}_{\boldsymbol{x}}}\left(\boldsymbol{\mu}_{\boldsymbol{x}}^{k+1} - \boldsymbol{\mu}_{\boldsymbol{x}}^{k}\right) \\ &= \boldsymbol{x}_{IMPP}^{k} + \frac{\partial h^k}{\partial \boldsymbol{\mu}_{\boldsymbol{x}}}\left(\boldsymbol{\mu}_{\boldsymbol{x}}^{k+1} - \boldsymbol{\mu}_{\boldsymbol{x}}^{k}\right) \end{aligned} . \quad (20)$$

With our approach, h is approximated in the form of $\bar{h}$, see Eqn. (21). The $\bar{h}$ is defined in Eqn. (19) based on the linearized limit state function. With this approximation, the partial derivatives in Eqn. (20) can be derived from the analytical expression of $\bar{h}$ instead of the unknown function h. We call this method "Quasi First Order Approximation" because it is not necessary to evaluate the first-order derivatives in Eqn. (20) numerically.

$$\boldsymbol{x}_{IMPP}^{k+1} \approx \boldsymbol{x}_{IMPP}^{k} + \frac{\partial \bar{h}^k}{\partial \boldsymbol{\mu}_{\boldsymbol{x}}}\left(\boldsymbol{\mu}_{\boldsymbol{x}}^{k+1} - \boldsymbol{\mu}_{\boldsymbol{x}}^{k}\right). \quad (21)$$

With the analytical approach, the i^{th} component of $\bar{h}$ is $\bar{h}_i$ in Eqn. (19). In the case of $\sigma_{x_i} = r_i \mu_{x_i}$, where the coefficients of variation r_i are constants, the partial derivatives can be evaluated as:

For $i = j$

$$\frac{\partial \bar{h}_i^k}{\partial \mu_{x_j}} = -1 + \frac{2X_{IMPPi}^k}{\mu_{x_i}} + \frac{\left(\mu_{x_i} - X_{IMPPi}^k\right)^3 a_i^2 r_i^2}{\left(a_i \beta\, r_i^2\right)^2 \mu_{x_i}^3}\Bigg|_{\mu_{x_i} = \mu_{x_i}^k} \quad (22)$$

For $i \neq j$

$$\frac{\partial \bar{h}_i^k}{\partial \mu_{x_j}} = \frac{a_j^2 r_j^2 \mu_{x_j}\left(\mu_{x_i} - X_{IMPPi}^k\right)^3}{\left(a_i \beta\, r_i^2\right)^2 \mu_{x_i}^4}\Bigg|_{\mu_{x_i} = \mu_{x_i}^k,\, \mu_{x_j} = \mu_{x_j}^k} \quad (23)$$

Note that if the partial derivative is close to one (i.e., the change of IMPP in $\boldsymbol{x}$-domain is the same as the change of $\boldsymbol{\mu_x}$ from cycle k to k+1), the above equation of $\boldsymbol{x}_{IMPP}^{k+1}$ in Eqn. (21) degenerates into the expression used in the original SORA as in Eqn. (9). Similar to Approach 2, $\boldsymbol{b}$ is obtained by linearizing the $g(\boldsymbol{x})$ function at the exact IMPP from RA in cycle k without the need for additional function evaluations. For a random design variable X_i with constant variance, the partial derivatives in Eqns. (22) and (23) are taken as one and zero, respectively. Under such condition, the IMPP estimation formulation is the same as the one used in the original SORA.

Because the Taylor expansion is only a local expansion in a small neighborhood of the expanding point, we expect that the estimation will not be very accurate in the first several cycles when $\boldsymbol{\mu_x}$ may change significantly. The estimation will become more accurate in later cycles when design points do not differ too much.

If we consider problems with constant variances as a special case of using all four approaches presented, we expect that all the methods except Approach 2 should behave the same due to the same estimation formulation of IMPP when taking the variances as constants. Approach 2 differs slightly since its estimation of IMPP is based on linearized constraints. In Section 4, all these four approaches are applied in several examples to check their validity and effectiveness.

3.5 IMPLEMENTATION FOR DIFFERENT TYPES OF DISTRIBUTIONS

In the previous sections that introduce the different proposed formulations, we have assumed that $\boldsymbol{x}$ follow normal distributions which are commonly used in engineering applications. For problems with non-normal distributions, the same strategies for predicting an IMPP in a new cycle can be followed, but the μ_x and $\sigma_x(\mu_x)$ in those formulations need to be updated to μ_x^N and σ_x^N, the descriptors of mean and standard deviation for equivalent normal distributions. Following Rackwitz-Fiessler's two parameter equivalent normal method [27], the μ_x^N and σ_x^N of an equivalent normal distribution at a point of interest X^* can be expressed as,

$$\begin{aligned} \sigma_x^N &= \phi\left\{\Phi^{-1}\left[F_x\left(X^*\right)\right]\right\} / f_x\left(X^*\right) \\ \mu_x^N &= X^* - \Phi^{-1}\left[F_x\left(X^*\right)\right]\sigma_x^N \end{aligned} , \quad (24)$$

where ϕ is the Probability Density Function (PDF) of standard normal distribution and F_x, f_x are the CDF and PDF of the non-normal distribution of X, respectively. The same transformation needs to be used for searching the IMPP in the RA that follows the DO in each cycle.

4. COMPARATIVE STUDIES

In this section, two mathematical problems and one design example are used to compare the effectiveness of the three proposed formulations as well as the original SORA approach (approach 0). The results from the Double Loop Method using the R-percentile formulation are considered as the "exact" solutions for reference. All random variables are assumed to follow normal distributions with $\sigma_{x_i} = r_i \mu_{x_i}$, where r_i is the constant coefficient of variation for X_i.

4.1 PROBLEM 1 WITH LINEAR CONSTRAINTS

We consider only linear constraints in the first mathematical example as shown in Eqn. (25), where the objective function is nonlinear and all four constraint functions g's are linear in the original design space. All input variables are random design variables (X_1 to X_6) with normal distributions $N(\mu_i, \sigma_i)$, $i = 1, \cdots, 6$. The desired reliability is 0.99865 for each constraint. For each approach, we examine two cases with different values of coefficients of variation and two other cases with different constant variances.

Table 1 illustrates the efficiency and accuracy of the various approaches considered for changing variances. With a numerical tolerance of 1%, we find that all approaches reach the same optimal solution for both cases. All three proposed approaches (1 to 3) improve the efficiency greatly compared to the Double Loop method (DLP). The three proposed approaches also achieve better performance compared to the original SORA method (Approach 0). It is noted that the magnitude of coefficients of variations, $\boldsymbol{r}_x$, has some impact on the efficiency of all methods except Approach 2, which is shown to be the most efficient in all cases. This is reasonable because Approach 2 predicts the new IMPP by linearizing the limit state function. Since the original constraints are all linear, Approach 2 requires at most 3 cycles (including one cycle for convergence check) to find the final optimal solution no matter how large the variance is. On the other hand, Approaches 1 and 3 are just as efficient as Approach 2 when the coefficient of variation is small; but require more cycles when the coefficient becomes larger. The original SORA method and the DLP method generally requires more function calls when the coefficient becomes larger.

$$
\begin{aligned}
&\min \quad f(\boldsymbol{x}) = \frac{\mu_1\mu_2 - \mu_4^2}{\mu_3} - \sqrt{\mu_5}\mu_6^3 \\
&\text{s.t.} \quad P(g_i(\boldsymbol{x}) \ge 0) \ge R_i \qquad i = 1,\cdots,4 \\
&\quad g_1(\boldsymbol{x}) = -X_1 + 3X_2 - 5 \ge 0 \\
&\quad g_2(\boldsymbol{x}) = -X_1 - 2X_3 - X_6 + 10 \ge 0 \\
&\quad g_3(\boldsymbol{x}) = X_1 + 2X_4 - X_5 - 8 \ge 0 \\
&\quad g_4(\boldsymbol{x}) = X_2 - 7X_6 + 2 \ge 0 \\
&\quad 1 \le \mu_1 \le 10 \qquad 2 \le \mu_2 \le 8 \\
&\quad 3 \le \mu_3 \le 8 \qquad 3 \le \mu_4 \le 8 \\
&\quad 1 \le \mu_5 \le 6 \qquad 0.1 \le \mu_6 \le 2
\end{aligned} \quad . \quad (25)
$$

Table 1. Comparison for Example 1 ($\sigma_i = 0.02\mu_i$ and $\sigma_i = 0.15\mu_i$)

r_i	*Approaches*	*x*	*f*	*Cyc.*	*F.C.*
0.02	**Approach 0**	***x***=[1.0000, 8.0000, 3.0000, 8.0000, 6.0000, 1.3236]	-24.3472	5	240
	Approach 1			3	187
	Approach 2			3	180
	Approach 3			3	173
	Double Loop			N/A	951
0.15	**Approach 0**	***x***=[1.0000, 3.6479, 3.0000, 8.0000, 1.7440, 0.2603]	-20.1406	6	318
	Approach 1			4	235
	Approach 2			3	198
	Approach 3			4	235
	Double Loop			N/A	1084

Cyc. ----- Cycles; ***F.C.*** ----- Function Call

Table 2. Comparison for Example 1 (constant variation)

	$\sigma_i = 0.02$		$\sigma_i = 0.15$	
Approach	***Cycles***	***F.C.***	***Cycles***	***F. C.***
0	3	156	3	170
1				170
2				170
3				172
DLP	N/A	828	N/A	2918

The results of two cases with constant variances are presented in Table 2. Except the DLP method, all approaches converge to the same final optimal solution at Cycle 3. For larger variance, because all constraints are linear, the number of cycle doesn't increase for all approaches. On the other hand, the numbers of function calls increase in almost the similar amount because these additional function evaluations are all used for IMPP search in RA.

4.2 PROBLEM 2 WITH NONLINEAR CONSTRAINTS

Problem 2 is developed from the example introduced in [15] with two random design variables X_1, X_2. Both of them are normally distributed with means of μ_1, μ_2 and deviations σ_1, σ_2 respectively. The problem has three nonlinear constraints with the same desired reliability of 0.99865. The formulation used by [15] is modified slightly here in the first constraint, which helps enlarge the feasible design region to explore the behaviors of our proposed approaches for large changing variances. The results with only two design variables facilitate the graphical illustration to help better understand the proposed approaches. Problem 2 is again examined for two cases with different values of coefficients of variation. The comparison of results from different approaches is provided in Table 3.

$$
\begin{aligned}
&\min \quad f(\boldsymbol{x}) = \mu_1 + \mu_2 \\
&\text{s.t.} \quad P(g_i(\boldsymbol{x}) \ge 0) \ge R_i \\
&\quad g_1(\boldsymbol{x}) = X_1^2 + \frac{X_2}{20} - 1 \ge 0 \\
&\quad g_2(\boldsymbol{x}) = \frac{(X_1 + X_2 - 5)^2}{30} + \frac{(X_1 - X_2 - 12)^2}{120} - 1 \ge 0 \\
&\quad g_3(\boldsymbol{x}) = \frac{80}{X_1^2 + 8X_2 + 5} - 1 \ge 0 \\
&\quad 0.01 \le \mu_i \le 10, \quad i = 1, 2
\end{aligned} \quad . \quad (26)
$$

Table 3. Results of Example 2 for constant variances

$\sigma_i = \text{const.}$	$\sigma_i = 0.02$		$\sigma_i = 0.20$		$\sigma_i = 0.40$	
Approach	***Cyc.***	***F.C.***	***Cyc.***	***F.C.***	***Cyc.***	***F.C.***
0	3	77	3	107	4	279
1						279
2						**276**
3						279
DLP	N/A	201	N/A	288	N/A	490

Table 4. Results of Example 2 for changing variances

$\sigma_i = r_i\mu_i$	$r_i = 0.02$		$r_i = 0.10$		$r_i = 0.20$	
Approach	***Cyc.***	***F.C.***	***Cyc.***	***F.C.***	***Cyc.***	***F.C.***
0	6	119	13	241	18	728
1	3	71	3	77	6	347
2					**4**	**250**
3					5	306
DLP	N/A	201	N/A	237	N/A	372

Within the numerical tolerance of 0.02%, all approaches generate the same optimal results for constant and changing variances. From the results listed in Tables 3 and 4, it is noted that for all approaches, it generally needs more cycles and function calls to reach the final optimal solutions for cases with larger constant variances or larger coefficients of variation. For problems with constant variance, all proposed approaches as well as the original SORA method are more efficient than the Double Loop Method. While dealing with changing variance, the original SORA method is the most sensitive to the coefficient of variation. It requires much more cycles and function calls compared to the three proposed approaches when the coefficient becomes larger. For problems with changing variance, because the variance is location-dependent, the exact IMPP becomes more sensitive to the location of a design, posing challenges for IMPP estimation. Among all approaches, Approach 2 is shown to be the most effective in all cases. In Fig. 5, we graphically illustrate the locations of the predicted (estimated) IMPPs and the exact IMPPs from cycle to cycle in the decoupled single loop process with the formulation used with Approach 2, for ***r*** = [0.2 0.2]. The IMPPs illustrated are only associated with g_2, an active constraint at the optimal solution.

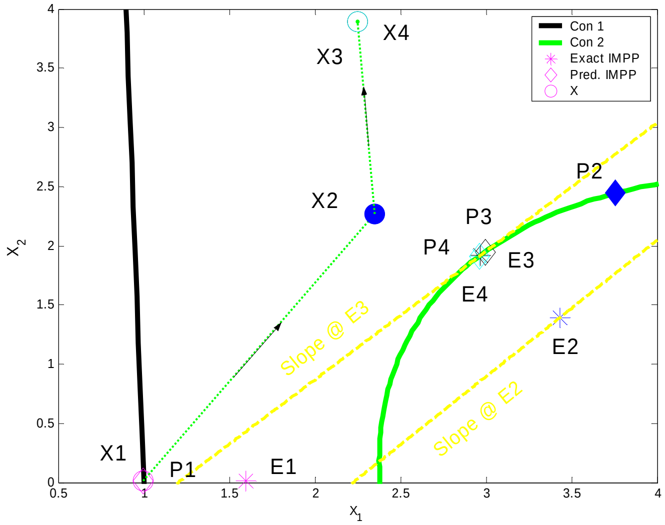

Figure 5. History of predicted and exact IMPPs with Approach 2 (Example 2)

The numbers 1 to 4 in Fig. 5 denote the cycle numbers from 1 to 4. X, P and E refer to the design point (obtained from DO), predicted IMPP (using Approach 2), and the exact IMPP (verified by RA), respectively. The feasible region is between the boundaries of constraints 1 and 2. In the first cycle, since no information of IMPP exists, the IMPP is set as the design point. The exact IMPP (E1) verified from the RA shows that E1 deviates from P1. The information of E1 is used to predict the IMPP in the next cycle (P2). It shows that P2 sits exactly on the boundary of equivalent deterministic constraint 2. However, after RA, it is found that the exact IMPP (E2) locates in the infeasible region. In Cycle 3, the information of E2 is used to predict P3. It shows that the predicted P3 is quite close to the exact one (E3), almost overlapping with each other on the constraint boundary. Cycle 3 brings an infeasible solution back to a feasible one. The final cycle (cycle 4) provides the convergence check of the optimal result. The locations of P4 and E4 almost overlap with those of P3 and E3. From this illustration, we note that Approach 2 works quite effectively even though the locations of design point X vary quite a lot and the variance is large from cycle to cycle. From the plot of the slopes of the constraint functions at the exact IMMPs in cycles 2 and 3, i.e., E2 and E3, respectively (dash lines in Fig. 5), we see that the slopes at these two IMMPs are almost identical even though the design point has moved quite a lot. This matches with the assumption used in Approach 2 for IMPP prediction and results in good estimations.

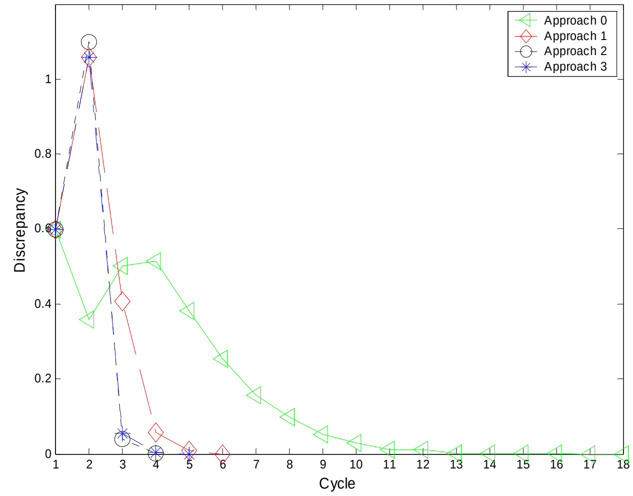

Figure 6. Discrepancy of the estimated and exact IMPPs $(r_i = 0.20)$

To compare the effectiveness of the IMPP estimation using different approaches, the discrepancies of the estimated IMPPs and the exact IMPPs are plotted in Fig. 6. All the discrepancies are described in terms of the distances between the estimated and the exact IMPPs. Discrepancies of all approaches in the first cycle are the same because the IMPP is initially set as the design point and the DO produces the same result. All discrepancies go down to zero with the increase of cycles, indicating all approaches are able to converge and to finally match the estimated and the exact IMPPs despite the different cycles needed. Compared to the original SORA approach (Approach 0), Approaches 1, 2 and 3 have significantly improved the efficiency. There is a little oscillation in the curve of Approach 0 before the discrepancy vanishes. That is because the shifting vector used in the original SORA method does not bring the change of design point and the effect of changing variance into account. In the first several cycles, because the design points differ a lot compared to the later cycles, the effect of changing variance causes relatively larger discrepancies in prediction. Approaches 2 and 3 provide better estimations of the IMPP compared to Approach 1. Overall, Approach 2 is shown to be the most efficient.

4.3 A VESSEL DESIGN EXAMPLE

In this example, the task is to design a specified type of gas vessel working under a given pressure. The problem formulation (Eqn. (27)) is derived from the example used in [28]. The optimization problem is to find the wall thickness t, inner radius R and the mid-section length L, all considered as normally distributed random variables with changing variance due to the randomness in the manufacturing process. The design objective is to obtain the maximum volume, while avoiding the yielding of material in both the circumferential and radial directions when loaded with an internal pressure P, a random parameter with constant deviation of 10 lbs. Geometric constraints are specified as a part of the problem. The material is given as UNS G101000 HR, with a yielding strength of 26,000 psi. Due to the nonidentical properties of the material, the effective yielding stress Y is considered as a random parameter with the mean as the half of the nominal yielding stress; the variance is given as a constant 260 psi.

$$
\begin{aligned}
&\text{find} \quad \mu_t, \mu_R, \mu_L \\
&\text{Max} \quad f(\boldsymbol{x}) = \frac{4\pi R^3}{3} + \pi R^2 L \\
&\text{s.t.} \quad P(g_i(\boldsymbol{x}) \ge 0) \ge R_i, \quad i = 1, \cdots, 5 \\
&\quad g_1(\boldsymbol{x}) = 1.0 - \frac{P(R+0.5t)/(2t)}{Y/2} \ge 0 \\
&\quad g_2(\boldsymbol{x}) = 1.0 - \frac{P(2R^2 + 2Rt + t^2)/(2Rt + t^2)}{Y/2} \ge 0 \\
&\quad g_3(\boldsymbol{x}) = 1.0 - \frac{L + 2R + 2t}{60} \ge 0 \\
&\quad g_4(\boldsymbol{x}) = 1.0 - \frac{R+t}{12} \ge 0 \\
&\quad g_5(\boldsymbol{x}) = \frac{5t}{R} - 1.0 \ge 0 \\
&\quad 0.25 \le t \le 2.0 \quad \mu_P = 1000 \quad \text{psi} \\
&\quad 6.0 \le R \le 24.0 \quad \mu_Y = 260000 \,\text{psi} \\
&\quad 10.0 \le L \le 48.0
\end{aligned} \tag{27}
$$

Various approaches are used to solve the problem by assuming that the variances of all three design variables $\boldsymbol{x} = [t \;\; R \;\; L]$ are changing with the coefficients $\boldsymbol{r}$ = [0.2 0.2 0.1].

Table 5. Comparison of approaches for Example 3

	[t, R, L, f] = [2.0000 6.1398 36.383 5278.3]				
Approach	**0**	**1**	**2**	**3**	**DLP**
Cycles	7	4	4	4	N/A
Fun. Call	432	293	306	316	1481

Table 5 lists the comparison of results for R= 99.865%, set for all constraints. All the single loop approaches require much less cycles to reach the same optimal solution than the double loop method. The three proposed approaches are more efficient than the original SORA method, even though it is not clear among these three approaches which one is superior to the others for this example problem. Because the problem is nonlinear, it takes all approaches more than 3 cycles to reach the optimal solution.

In the vessel design example, we considered a random design parameter, the internal pressure P, that has constant mean and deviation. When implementing the proposed approaches, the random parameters can be treated as special cases of random design variables with constant means and fixed variances. Based on the formulations of different approaches, we find that except approach 2, all other approaches set the IMPPs of random parameters in a new cycle as the exact IMPPs from the previous cycle, i.e. $\boldsymbol{p}_{IMPP}^{k+1} \approx \boldsymbol{p}_{IMPP}^{k}$. In our implementation, to focus on the comparision of $\boldsymbol{x}_{IMPP}$ estimations, Approach 2 employs the same approximation for IMPPs of random parameters as the other approaches.

5. CONCLUSIONS

In this paper, to accommodate the effect of changing variance in probabilistic design, three different approaches are developed to enhance the SORA method. These methods share the common strategy as the original SORA method under which the double loop probabilistic optimization is decoupled into cycles of single loop deterministic optimization (DO) followed by reliability assessment (RA). The three new approaches deviate in the formulation of the DO, in particular in the formulation for predicting the IMPP in a new cycle. By examining the mathematical principles and through empirical studies, we find that our proposed approaches inherit the high efficiency of the original SORA, while improving the efficiency of solving probabilistic optimization problems with changing variance.

The three new approaches are distinguished by the different strategies of IMPP estimations. Among them, the Approximated $\boldsymbol{u}_{IMPP}$ Approach (Approach 1) approximates the IMPP at a new cycle k+1 by replacing $\boldsymbol{u}_{IMPP}^{k+1}$ in the IMPP equation by $\boldsymbol{u}_{IMPP}^{k}$ and updating the mean and variance information based on the location of the new design point. Since the IMPP in the $\boldsymbol{u}$-space often changes when the design point changes, this estimation is quite rough. Approach 1 is shown to be the least efficient among the three new approaches.

The Direct Linear Estimation Formulation (Approach 2) provides the estimation of the IMPP in a new cycle using the slope information of the linearized limit state function expanded at the exact IMPP from the previous cycle, in addition to updating the mean and variance information at the new design point. Approach 2 is shown to be the most efficient method in the examples tested, especially when the changing variance is large. For problems with all active constraints being linear, it takes Approach 2 three cycles to converge (including convergence check) no matter the magnitude of variance. Approach 2 also has excellent behavior for nonlinear constraints as illustrated in this paper. Through the empirical study, we find that Approach 2 works effectively because in most applications the slope of the limit state function does not change too much even when the design point varies. This is the reason why Approach 2 provides a good estimation of a new IMPP.

The Quasi First Order Approximation of $\boldsymbol{x}_{IMPP}$ Formulation (Approach 3) provides the estimation of an IMPP by employing the first-order Taylor expansion at the exact IMPP from the previous cycle. The derivative information is obtained by linearizing the limit state function; the mean and variance information are also updated at the new design point. Because Taylor expansion is only a local expansion in a small neighborhood of the expanding point, the estimation is shown to be less accurate in the first several cycles (when the design point varies significantly) compared to the later cycles (when solution converges). Our empirical study shows that the overall efficiency of Approach 3 is the second among the tested approaches. The accuracy of Approach 3 can be further improved by introducing high-order derivatives in Taylor expansion, a topic for future work.

In conclusion, our study further illustrates that the strategy used in SORA for decoupling deterministic optimization and reliability assessment is an effective approach for improving the accuracy in IMPP estimation because a new IMPP estimation is always based on the exact IMPP from the previous cycle. All three proposed approaches overcome the limitation of the original SORA method by incorporating the information of changing variance. Among them, the Direct Linear Estimation Formulation (Approach 2) is shown to be the most effective for the problems tested. The insight gained from our study on the applicability of different approaches can be extended and utilize in other probabilistic optimization strategies that require MPP or IMPP estimations.

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