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COLLABORATIVE RELIABILITY ANALYSIS FOR MULTIDISCIPLINARY SYSTEMS DESIGN

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Abstract

Traditional Multidisciplinary Design Optimization (MDO) generates deterministic optimal designs, which are frequently pushed to the limits of design constraint boundaries, leaving little or no room to accommodate uncertainties in system input, modeling, and simulation. As a result, the design solution obtained may be highly sensitive to the variations of system input which will lead to performance loss and the solution is often risky (high likelihood of undesired events). Reliability-based design is one of the alternative techniques for design under uncertainty. The natural method to perform reliability analysis in multidisciplinary systems is the all-in-one approach where the existing reliability analysis techniques are applied directly to the system-level multidisciplinary analysis. However, the all-on-one reliability analysis method requires a double loop procedure and therefore is generally very time consuming. To improve the efficiency of reliability analysis under the MDO framework, a collaborative reliability analysis method is proposed in this paper. The procedure of the traditional Most Probable Point (MPP) based reliability analysis method is combined with the collaborative disciplinary analyses to automatically satisfy the interdisciplinary consistency in reliability analysis. As a result, only a single loop procedure is required and all the computations are conducted concurrently at the individual discipline-level. Compared with the existing reliability analysis methods in MDO, the proposed method is more efficient and therefore provides a cheaper tool to evaluate design feasibility in MDO under uncertainty. Two examples are used for the purpose of verification.

1. Introduction

Multidisciplinary Design Optimization (MDO)[1] has become a systematic approach to the optimization of complex, often coupled engineering systems. Here, "multidisciplinary" refers to the different aspects that must be included in designing a system that involves multiple interacting disciplines, such as those found in aircraft, spacecraft, automobiles, and industrial manufacturing applications. Numerous successful examples of MDO applications have been found in many areas, such as Electromagnetics[2], High Speed Civil Transport Design[3, 4], Space Vehicle Design[5], Aerospike Nozzle Design[6], Rotor Design[7, 8], Integrated Controls-Structures Design[9], Integrated Circuit Design[10], and Automobile Design[11].

However, the traditional MDO generates deterministic optimal designs, which are frequently pushed to the limits of design constraint boundaries, leaving little or no room for accommodating uncertainties in system input, modeling, and simulation. As a result, the design solution obtained may be 1) highly sensitive to the variation of system input which will lead to performance loss and risky (high likelihood of undesired events), or 2) conservative and therefore uneconomic if the deterministic safety factors are utilized.

To overcome the drawbacks of deterministic MDO, techniques for uncertainty analysis under the MDO framework have been proposed and have been getting much attention[12]. In recent developments, some preliminary results of multidisciplinary design under uncertainty are reported[12-17]. In these works, the mean and variance of system performance are evaluated through uncertainty analysis and then utilized to obtain optimal solutions based on robustness considerations. For example, in Du and Chen's work[12,16], the system uncertainty analysis (SUA) and the concurrent subsystem uncertainty analysis (CSSUA) methods are proposed to evaluate performance variances taking into account the multidisciplinary design framework. In Gu's work[17], the "worst case" concept and the first-order sensitivity analysis are used to evaluate the interval of the end performance of a multidisciplinary system. Even though the mean, the variance, and the interval of system performance are sufficient to evaluate the robustness of a design objective, they are generally not rigorous to be used for formulating the design feasibility (design constraints) under uncertainty. The ideal formulation of the design feasibility under uncertainty is the use of probabilistic constraints or the so-called reliability-based constraints wherein the design feasibility is modeled by the probability of constraint satisfaction (reliability)[18] and wherein the complete shape of a performance distribution, especially that at the tail, is taken into account.

Recently, much attention has been turned to the development of procedures to couple reliability analysis and MDO[19-21]. In the work of Sues[19], response surface models of system output are created at the system level to replace the computationally expensive simulation models. Using the response surface models, reliability analysis is conducted for MDO under uncertainty. The drawback of using this approach is the cost associated with generating an accurate response surface model over a large parameter space (for both deterministic and random variables). Besides, some of the response surface methods tend to "smooth" a performance behavior.

A framework for reliability-based MDO was proposed in Ref. 20. In their work, the reliability analysis is decoupled from the optimization. Reliabilities are computed initially before the first execution of the optimization loop, and then updated after the optimization loop is executed. However, in the optimization loop, approximate forms of probabilistic constraints are used. To integrate the existing reliability analysis techniques into the MOD framework, a more tightly, multi-stage, and parallel implementation strategy of probabilistic design optimization was utilized by Koch, et al. [21]. Nevertheless, in all these existing frameworks, most computations are spent on the reliability analysis during the optimization process. The efficiency of reliability analysis dominates the overall efficiency of the whole design process. Since the reliability analysis in these design frameworks is usually conducted based on the system-level multidisciplinary analysis, as we will see next, two loops of iterative computations will be involved and as a result, MDO under uncertainty becomes much less affordable compared to deterministic MDO.

To improve the efficiency of reliability analysis for MDO and eventually MDO under uncertainty, a collaborative reliability analysis method is proposed in this paper. In this method, the procedure of the traditional Most Probable Point (MPP) based reliability analysis method is combined with the collaborative disciplinary analyses to automatically satisfy the interdisciplinary consistency in reliability analysis. As a result, only a single loop procedure is required and all the computations are conducted concurrently at the individual discipline-level.

The paper is organized as follows. The general multidisciplinary system analysis is reviewed in Section 2. In Section 3, the strategy of the traditional all-in-one reliability analysis for multidisciplinary systems is discussed and the large computational needs of this approach are highlighted. Our proposed collaborative reliability analysis method for multidisciplinary systems design is presented in Section 4 and two examples are used to illustrate the effectiveness of the proposed method in Section 5. Section 6 is the closure which highlights the effectiveness of the proposed method and provides discussions on its applicability under different circumstances.

2. The Multidisciplinary System

For simplicity, we use a 3-discipline system to present the method. The conclusions drawn based on the 3-discipline system can be easily generalized for an n-discipline system. Fig. 1 shows the 3-discipline system, where each box represents the analysis (simulation) that belongs to a discipline. $\mathbf{x}_s$ are the system input variables which are the input for all disciplines, also called sharing variables. $\mathbf{x}_i$ (i = 1, 2, and 3) are the input variables of discipline i. $\mathbf{x}_s$ and $\mathbf{x}_i$ are mutually exclusive sets. Note that in this paper, the bold font stands for a vector and a regular font stands for a scalar variable. Therefore, **x** represents a vector and x represents a variable or an element of vector **x**. In some circumstances, a bold font also represents a function vector as we will see later on. $\mathbf{y}_{ij}(i \neq j)$ are interdisciplinary linking variables, which are those functional outputs calculated in discipline i , at the same time, are required as inputs to discipline j. $\mathbf{z}_i$ are outputs of discipline i.

For discipline 1, the disciplinary input-output relations have the functional form

$$\mathbf{z}_1 = \mathbf{F}_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{21}, \mathbf{y}_{31}) \tag{1}$$

$$\mathbf{y}_{12} = \mathbf{F}_{y12}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{21}, \mathbf{y}_{31}) \tag{2}$$

$$\mathbf{y}_{13} = \mathbf{F}_{y13}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{21}, \mathbf{y}_{31}) \tag{3}$$

Similarly, for disciplines 2 and 3, we have the disciplinary input-output relations

$$\mathbf{z}_2 = \mathbf{F}_{z2}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_{12}, \mathbf{y}_{32}) \tag{4}$$

$$\mathbf{y}_{21} = \mathbf{F}_{y21}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_{12}, \mathbf{y}_{32}) \tag{5}$$

$$\mathbf{y}_{23} = \mathbf{F}_{y23}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_{12}, \mathbf{y}_{32}) \tag{6}$$

and

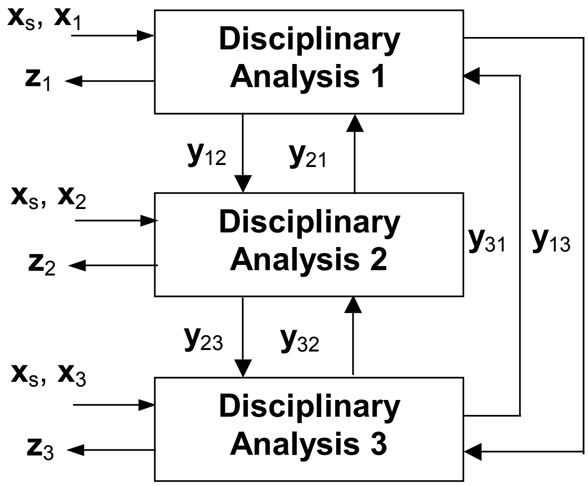

Figure 1. A Multidisciplinary System

$$\mathbf{z}_3 = \mathbf{F}_{z3}(\mathbf{x}_s, \mathbf{x}_3, \mathbf{y}_{13}, \mathbf{y}_{23}) \quad (7)$$

$$\mathbf{y}_{31} = \mathbf{F}_{y31}(\mathbf{x}_s, \mathbf{x}_3, \mathbf{y}_{13}, \mathbf{y}_{23}) \quad (8)$$

$$\mathbf{y}_{32} = \mathbf{F}_{y32}(\mathbf{x}_s, \mathbf{x}_3, \mathbf{y}_{13}, \mathbf{y}_{23}) \quad (9)$$

The disciplinary analysis F maps disciplinary input into disciplinary output. F can be of analytical forms or black boxes of simulation tools. **F** are assumed to be independently solvable. Taking $\mathbf{F}_{z1}$ as an example, given appropriate inputs $(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{21}, \mathbf{y}_{31})$ for which the analysis is defined, we can compute the disciplinary output $\mathbf{z}_1$ through disciplinary 1 analysis $\mathbf{z}_1 = \mathbf{F}_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{21}, \mathbf{y}_{31})$.

The coupled multidisciplinary analysis system depicted in Fig. 1 reflects the physical requirement that a solution simultaneously satisfy the three disciplinary analyses[22]. We write the multidisciplinary analysis system as a simultaneous system of equations as

$$\begin{cases} \mathbf{y}_{12} = \mathbf{F}_{y12}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{21}, \mathbf{y}_{31}) \\ \mathbf{y}_{13} = \mathbf{F}_{y13}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{21}, \mathbf{y}_{31}) \\ \mathbf{y}_{21} = \mathbf{F}_{y21}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_{12}, \mathbf{y}_{32}) \\ \mathbf{y}_{23} = \mathbf{F}_{y23}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_{12}, \mathbf{y}_{32}) \\ \mathbf{y}_{31} = \mathbf{F}_{y31}(\mathbf{x}_s, \mathbf{x}_3, \mathbf{y}_{13}, \mathbf{y}_{23}) \\ \mathbf{y}_{32} = \mathbf{F}_{y32}(\mathbf{x}_s, \mathbf{x}_3, \mathbf{y}_{13}, \mathbf{y}_{23}) \end{cases} \quad (10)$$

Solving the coupled equations (10) leads to a full multidisciplinary analysis and we call this analysis the system-level multidisciplinary analysis, or simply system-level analysis, in which the coupled disciplines give a physically consistent result.

Without the consideration of uncertainty, a general MDO model is simplified as:

$$\begin{aligned} \min \quad & f(\mathbf{x}_s, \mathbf{z}_1^{'}, \mathbf{z}_2^{'}, \mathbf{z}_3^{'}) \\ s.t. \quad & \mathbf{z}_1^{''}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{21}, \mathbf{y}_{31}) \ge 0 \\ & \mathbf{z}_2^{''}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_{12}, \mathbf{y}_{32}) \ge 0 \\ & \mathbf{z}_3^{''}(\mathbf{x}_s, \mathbf{x}_3, \mathbf{y}_{13}, \mathbf{y}_{23}) \ge 0 \end{aligned} \quad (11)$$

where f is the collaborative design objective, representing the function of system design variables xs and subsystem performance $\mathbf{z}_\mathrm{i}^{'}\ (\mathrm{i} = 1, 2 \text{ and } 3)$ which are part of the output z of discipline i. $\mathbf{z}_\mathrm{i}^{''}\ (\mathrm{i} = 1, 2 \text{ and } 3)$ stand for those subsystem performance that are considered as design constraints.

In many engineering problems, randomness is associated with system input variables $\mathbf{x}_s$ and disciplinary input variables $\mathbf{x}_i$. Examples of the randomness include the random material properties, manufacturing tolerances, stochastic loads, and stochastic operation environments, that can be described by probabilistic distributions. Since the output $\mathbf{z}_i$ (i = 1, 2 and 3) are functions of random input variables $\mathbf{x}_s$ and disciplinary input variables $\mathbf{x}_i$, $\mathbf{z}_i = \{\mathbf{z}_i^{'}, \mathbf{z}_i^{''}\}$ are also random variables. For the same reason, all the linking variables y_{ij} are also random variables. This phenomenon rouses the issue of reliability which is concerned with the probability of the design feasibility for $\mathbf{z}_\mathrm{i}^{''} \ge 0$.

3. All-in-One Reliability Analysis Method for MDO

With the existence of uncertainty, the deterministic MDO model (11) is reformulated as

$$\begin{aligned} \min \quad & f(\mathbf{x}_s, \mathbf{z}_1^{'}, \mathbf{z}_2^{'}, \mathbf{z}_3^{'}) \\ s.t. \quad & \mathrm{P}\{\mathbf{z}_1^{''}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{21}, \mathbf{y}_{31}) \ge 0\} \ge \mathrm{P}_1 \\ & \mathrm{P}\{\mathbf{z}_2^{''}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_{12}, \mathbf{y}_{32}) \ge 0\} \ge \mathrm{P}_2 \\ & \mathrm{P}\{\mathbf{z}_3^{''}(\mathbf{x}_s, \mathbf{x}_3, \mathbf{y}_{13}, \mathbf{y}_{23}) \ge 0\} \ge \mathrm{P}_3 \end{aligned} \quad (12)$$

The design feasibility under uncertainty is represented probabilistically such that the probability of the constraint satisfaction $\mathbf{z}_\mathrm{i}^{''} \ge 0$ is greater than or equal to the desired probability P_i. The probability of the constraint satisfaction can also be called the reliability. As we will discuss next, the reliability assessment is a critical component of MDO under uncertainty that demands much more computational effort than deterministic MDO. Efficient reliability analysis methods are therefore needed to suit the need of MDO. To explain the all-in-one reliability analysis method, we need to first explain the concept of the Most Probable Point (MPP) method.

For simplicity of discussion, in this section we use z (a scalar) to represent any element of the disciplinary system output vector $\mathbf{z}_i''$, x to represent all the inputs of disciplinary analysis (including linking variables as the input of discipline i), and F to represent the disciplinary analysis corresponding to z. For example, if we are interested in the reliability associated with one element z_1 out of the disciplinary output vector $\mathbf{z}_1''$, we then use $z = z_1$, $\mathbf{x} = (\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{21}, \mathbf{y}_{31})$, and $F = F_{z1}$. Therefore, disciplinary output of interest has functional relationship $z = F(\mathbf{x})$. In the reliability field, $z = F(\mathbf{x})$ characterizes the function of a specific performance criterion z and is called a limit state function. The failure surface or the limit state is defined as $F(\mathbf{x}) = c$ or simply $F(\mathbf{x}) = 0$. This is the boundary between the safe and failure regions in the random variables space. When $F(\mathbf{x}) > 0$, the system (or the discipline) is considered safe and when $F(\mathbf{x}) < 0$, the system can no longer fulfill the function for which it was designed. Fig. 2 shows the limit state for a two dimensional problem.

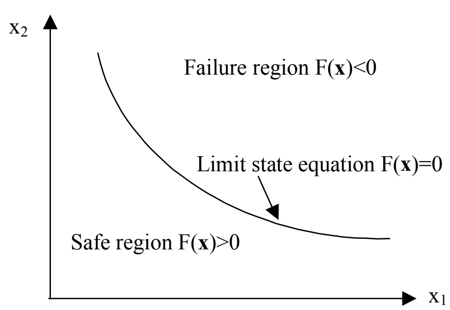

Figure 2. Limit State Concept

The probability of failure p_f is defined as the probability of the event that the system can no longer fulfill its function and p_f is given by

$$p_f = \mathrm{P}\{\mathrm{F}(\mathbf{x}) < 0\}, \tag{13}$$

which is generally calculated by the integral

$$p_f = \int \cdots \int_{F(\mathbf{x})<0} f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x}. \tag{14}$$

where $f_{\mathbf{x}}(\mathbf{x})$ is the joint probability density function (PDF) of $\mathbf{x}$ and the probability is evaluated by the multidimensional integration over the failure region $F(\mathbf{x})$.

The reliability R is the probability that the system functions properly and it is given by

$$R = P\{F(\mathbf{x}) > 0\} = 1 - p_f. \tag{15}$$

It is very difficult or even impossible to analytically compute the multidimensional integration in (14). An alternative method to evaluate the integration is Monte Carlo simulation[23]. However, when the probability of failure p_f is very small or the reliability is very high (close to 1), the computational effort of Monte Carlo Simulation is extremely expensive (this will be demonstrated by the examples in Section 5). To overcome this difficulty, Hasofer and Lind [24] proposed the concept of the Most Probable Point (MPP) to approximate the integration.

To make use of the MPP concept, the input random variables $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ (in the original design space, x–space) are transformed into an independent and standardized normal space $\mathbf{u} = \{u_1, u_2, \cdots, u_n\}$ (u–space). The most commonly used transformation is given by Rosenblatt [25] as

$$u_i = \Phi^{-1}[G_i(x_i)] \qquad (i = 1, \cdots, n), \tag{16}$$

where Φ^{-1} is the inverse of a normal distribution and G_i is the cumulative distribution function (CDF) of x_i. Eqn. (16) implies that the transformation maintains the CDFs being identical both in x-space and u-space.

The limit state function is now rewritten as

$$F(\mathbf{x}) = F(\mathbf{u}) = 0. \tag{17}$$

To easily assess reliability, Hasofer and Lind[24] used the safety index β which is defined as the shortest distance from the origin to a point on the limit-state surface in u-space (Fig. 3). Searching for β can be formulated as a minimization problem with an equality constraint:

$$\begin{cases} \beta = \min\limits_{\mathbf{u}} (\mathbf{u}^T \mathbf{u})^{1/2} \\ \text{subject to} \quad F(\mathbf{u}) = 0 \end{cases} \tag{18}$$

The solution of this minimization problem $\mathbf{u}_{MPP}$ is called the Most Probable Point (MPP). From Fig. 3, we see that the joint probability density function on the limit state surface has its highest value at the MPP and therefore the MPP has the property that in the standard normal space it has the highest probability of producing the value of limit state function $F(\mathbf{u})$ or the highest contribution to the integral (14) [26].

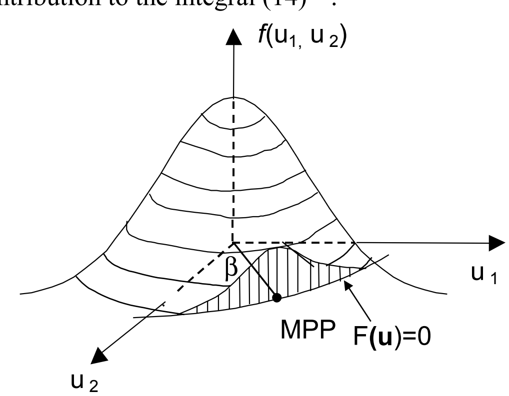

Figure 3. The MPP Concept

If the limit-state function $F(\mathbf{u})$ is linear, the accurate probability estimate at the limit state is given by the equation:

$$p_f = P\{F(\mathbf{x}) < 0\} = 1 - \Phi(\beta) . \quad (19)$$

The above equation provides an easy correspondence between the probability estimate and the safety index or the shortest distance β. Since (19) only utilizes the first order derivative of the limit state function, the method is called the First Order Reliability Method (FORM). Higher-order adjustments can be adopted if the magnitude of the principal curvatures of the limit-state surface in the u-space at the MPP is large[27]. Besides using optimization algorithms to solve problem (18), there exist many other MPP searching algorithms[26 - 30] developed in the field of structural reliability.

If the MPP based method applied directly to integrated multidisciplinary systems to evaluate the reliability, we call this approach all-in-one reliability analysis. In the following, we use one output of discipline 1, z_1, as an example to present the method and illustrate the huge computational effort associated with this approach. Here, we expect to evaluate the probability of failure (design feasibility) in discipline 1 and this is given by

$$p_f = P\{z_1 = F_{z1}(\mathbf{x}) < 0\} = P\{F_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{21}, \mathbf{y}_{31}) < 0\} \quad (20)$$

For a multidisciplinary system, since the distributions of inputs $\mathbf{y}_{21}$ and $\mathbf{y}_{31}$ (linking variables) are not known within the scope of discipline 1, we need to perform the system-level analysis to solve the linking variables $\mathbf{y}_{21}$ and $\mathbf{y}_{31}$, and eventually, the limit state function F_{z1} becomes the function of system inputs $(\mathbf{x}_s, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. Hence

$$p_f = P\{F_{z1}(\mathbf{x}) < 0\} = P\{F_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) < 0\} \quad (21)$$

Based on Eqn. (21), the mathematical model to find the MPP is formulated as

$$\begin{aligned} &\text{Minimize} \quad \beta = \left(\mathbf{u}^T\mathbf{u}\right)^{\frac{1}{2}} \\ &\text{DV=} \, \mathbf{u} = (\mathbf{u}_s, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \qquad (22) \\ &\text{Subject to } z_1 = F_{z1}(\mathbf{u}_s, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = 0 \\ &\text{DV - design variables} \end{aligned}$$

where $\mathbf{u}_s, \mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ are random variables in u-space corresponding to random design variables $\mathbf{x}_s, \mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ in x-space.

Due to the coupling nature of a multidisciplinary system, as illustrated in Fig. 4, there are two loops involved in solving the problem in (22) if an all-in-one approach is used. The outer loop is the minimization wherein the reliability index β is minimized (left box in Fig. 4) and the inner loop is the system-level analysis which is used to evaluate constraint function $z_1 = F_{z1}(\mathbf{u}_s, \mathbf{u}_1, \mathbf{u}_{21}, \mathbf{u}_{31})$ (right box in Fig.4). As discussed in Section 2, the system-level analysis is an iterative process where a simultaneous system of equations (10) is solved. Due to the close-loop condition, a number of individual disciplinary analyses are often required to solve a system of equations in order to achieve the compatibility between individual disciplines.

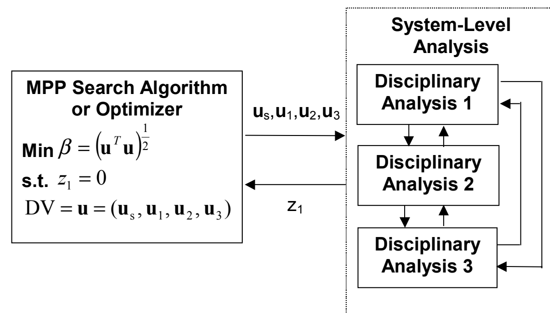

Figure 4. MPP Search Using the All-in-One Method

The advantage of the all-in-one reliability analysis is that it is easy to link the existing reliability analysis methods and computer programs to an all-in-one multidisciplinary system analysis. However, the efficiency of this method is not satisfactory since it needs many individual disciplinary analyses for system level convergence. To locate the MPP, the optimizer or the MPP search algorithm (outer loop) in Fig. 4 requires certain numbers of function evaluations for constraint function F_{z1} and each function evaluation of F_{z1} is one system-level analysis (inter loop) which requires a number of disciplinary analyses. As a result, the total number of individual disciplinary analyses can be very high. Suppose the number of function evaluations required by the outer loop optimization for MPP search is Nopt and the average total number of disciplinary analyses for each system-level multidisciplinary analysis is N_{disp}, the total number of disciplinary analyses N_{total} becomes

$$N_{total} = N_{opt} N_{disp} \quad (23)$$

To improve the efficiency of the reliability analysis for multidisciplinary systems, we propose a collaborative reliability analysis method which does not require any system-level analysis and significantly reduces the number of individual disciplinary analyses. The proposed method will be presented in detail in the next section and demonstrative examples will be given in Section 5.

4. The Collaborative Reliability Analysis for Multidisciplinary Systems

To reduce the total number of system level and subsystem level analyses, we use a single loop strategy. The optimization loop for MPP search and the system-level multidisciplinary analysis loop are combined to avoid the nested loops. The compatibility conditions for disciplines are formulated as constraint functions in the optimization model. By doing this, there is no need for maintaining the compatibility among disciplines in each function evaluation in the MPP search process as the all-in-one reliability analysis method. The compatibility will be achieved progressively in the process of the optimization for MPP search and will be satisfied eventually at the located MPP.

For the same problem presented in last section, the MPP searching problem is reformulated as

Minimize $\beta = \left(\mathbf{u}^T\mathbf{u}\right)^{\frac{1}{2}}$

DV= $\left\{\mathbf{u}, \mathbf{y}_{12}, \mathbf{y}_{13}, \mathbf{y}_{21}, \mathbf{y}_{23}, \mathbf{y}_{31}, \mathbf{y}_{32}\right\}$

and $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$

Subject to

$$
\begin{aligned}
& z_1 = F_{z1}(\mathbf{u}_s, \mathbf{u}_1, \mathbf{y}_{21}, \mathbf{y}_{31}) = 0 && \text{(DA1)} \\
& \mathbf{y}_{12} - \mathbf{F}_{y12}(\mathbf{u}_s, \mathbf{u}_1, \mathbf{y}_{21}, \mathbf{y}_{31}) = 0 && \text{(DA1)} \\
& \mathbf{y}_{13} - \mathbf{F}_{y13}(\mathbf{u}_s, \mathbf{u}_1, \mathbf{y}_{21}, \mathbf{y}_{31}) = 0 && \text{(DA1)} \\
& \mathbf{y}_{21} - \mathbf{F}_{y21}(\mathbf{u}_s, \mathbf{u}_2, \mathbf{y}_{12}, \mathbf{y}_{32}) = 0 && \text{(DA2)} \qquad (24) \\
& \mathbf{y}_{23} - \mathbf{F}_{y23}(\mathbf{u}_s, \mathbf{u}_2, \mathbf{y}_{12}, \mathbf{y}_{32}) = 0 && \text{(DA2)} \\
& \mathbf{y}_{31} - \mathbf{F}_{y31}(\mathbf{u}_s, \mathbf{u}_3, \mathbf{y}_{13}, \mathbf{y}_{23}) = 0 && \text{(DA3)} \\
& \mathbf{y}_{32} - \mathbf{F}_{y32}(\mathbf{u}_s, \mathbf{u}_3, \mathbf{y}_{13}, \mathbf{y}_{23}) = 0 && \text{(DA3)}
\end{aligned}
$$

DV - design variables

DA – Disciplinary analysis

The first equality constraint is the limit state function at its limit state. The remaining equality constraints stand for the interdisciplinary consistency conditions in (10). All the linking variables are also included as part of design variables. It should be noted that all the linking variables need to be transformed into the u-space. The proposed strategy is illustrated in Fig. 5 from which we see that the optimization for MPP search interacts with individual subsystem analyses separately but there are no direct interactions among subsystems. Taking discipline 1 as an example, the optimizer passes the design variables y_{21} and y_{31} (linking variables), as well as u_s (corresponding to system variables x_s) and u_1 (corresponding to disciplinary variables x_1) to discipline 1. The disciplinary analysis 1 is executed to compute a part of its outputs $\mathbf{F}_{y12}(\mathbf{u}_s, \mathbf{u}_1, \mathbf{y}_{21}, \mathbf{y}_{31})$ and $\mathbf{F}_{y13}(\mathbf{u}_s, \mathbf{u}_1, \mathbf{y}_{21}, \mathbf{y}_{31})$ which will serve as inputs of disciplines 2 (y_{12}) and 3 (y_{13}), respectively. To maintain the interdisciplinary compatibility, the equality constraints are set as $\mathbf{y}_{12} - \mathbf{F}_{y12}(\mathbf{u}_s, \mathbf{u}_1, \mathbf{y}_{21}, \mathbf{y}_{31}) = 0$ and $\mathbf{y}_{13} - \mathbf{F}_{y13}(\mathbf{u}_s, \mathbf{u}_1, \mathbf{y}_{21}, \mathbf{y}_{31}) = 0$ for discipline 1. Disciplines 2 and 3 work in the same way.

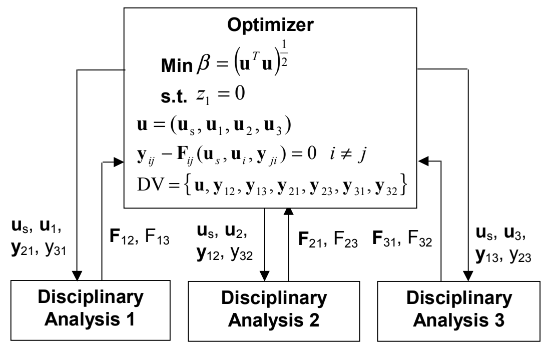

Figure 5. MPP Search in Collaborative Reliability Analysis

It is noted that with the proposed method, in the process of searching the MPP, only individual disciplinary analyses are required and no system-level multidisciplinary analysis is needed. All disciplinary analyses can be conducted concurrently which facilitates parellization. Since only one loop (the optimization loop for MPP search) is involved for iterative disciplinary analyses, compared with the all-in-one reliability analysis method, the collaborative reliability analysis method in general needs much less disciplinary analyses and hence is more efficient.

5. Examples

Two examples are used to illustrate the effectiveness of our proposed reliability analysis technique for multidisciplinary systems design. These two examples have been used in Ref. 16 to demonstrate the moment matching method for robust multidisciplinary design optimization where only the first two moments (the mean and the variance) of system performance are generated. We use these examples herein again for more rigorous formulation under uncertainty (reliability analysis). To verify our proposed method, we consider two aspects, namely, efficiency and accuracy. For efficiency, we compare the total number of individual disciplinary analyses needed for the proposed method with those for the all-in-one reliability analysis method. For accuracy, results from Monte Carlo Simulations with sufficient simulation sizes are considered as the reference solution for confirmation. The sequential quadratic programming (SQP) is used as the optimization search algorithm to locate the MPP in both collaborative reliability analysis and all-in-one reliability analysis.

Example 1

A multidisciplinary system is composed of two disciplines as shown in Fig. 6.

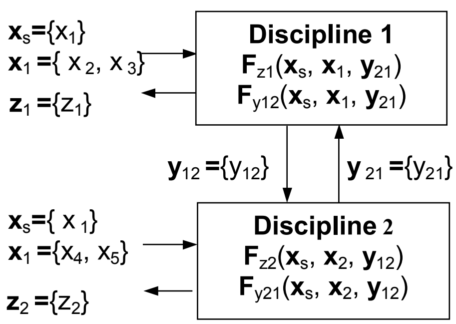

Figure 6. Example 1

For discipline 1, the functional relationships are represented as

$$\mathbf{x}_s = \{x_1\},\ \mathbf{x}_1 = \{x_2, x_3\},\ \mathbf{y}_1 = \mathbf{y}_{12} = \{y_{12}\},\ \mathbf{z}_1 = \{z_1\} \quad (25)$$

$$\begin{aligned}\mathbf{F}_{y12}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{21}) &= F_{y12}(x_1, x_2, x_3, y_{21}) \\ &= x_1^2 + 2x_2 - x_3 + 2\sqrt{y_{21}})\end{aligned} \quad (26)$$

$$\begin{aligned}\mathbf{F}_{z1}(\mathbf{x}_s, \mathbf{x}_1, \mathbf{y}_{21}) &= F_{z1}(x_1, x_2, x_3, y_{21}) \\ &= c - (x_1^2 + 2x_2 + x_3 + x_2 e^{-y_{21}})\end{aligned} \quad (27)$$

where c is a constant.

For discipline 2, the functional relationships are represented as

$$\mathbf{x}_s = \{x_1\},\ \mathbf{x}_2 = \{x_4, x_5\},\ \mathbf{y}_2 = \mathbf{y}_{21} = \{y_{21}\},\ \mathbf{z}_2 = \{z_2\} \quad (28)$$

$$\begin{aligned}\mathbf{F}_{y21}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_{12}) &= F_{y21}(x_1, x_4, x_5, y_{12}) \\ &= x_1 x_4 + x_4^2 + x_5 + y_{12}\end{aligned} \quad (29)$$

$$\begin{aligned}\mathbf{F}_{z2}(\mathbf{x}_s, \mathbf{x}_2, \mathbf{y}_{12}) &= F_{z2}(x_1, x_4, x_5, y_{12}) \\ &= \sqrt{x_1} + x_4 + x_5(0.4x_1)\end{aligned} \quad (30)$$

It is assumed that all the random variables x are normally distributed. The coefficient of variation (COV) of all the random variables is 0.1. The COV is the ratio of the standard deviation to the mean value.

Two design points are arbitrarily chosen for reliability analysis. At design point 1 where the mean values of $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ are $\boldsymbol{\mu}_x = (1,1,1,1,1)$, the limit state function is considered with c=5, namely

$$F_{z1} = 5 - (x_1^2 + 2x_2 + x_3 + x_2 e^{-y_{21}}) \quad (31)$$

The optimization problem for locating the MPP is formulated as follows

$$\begin{aligned}&\text{Minimize} \quad \beta = \left(\mathbf{u}^T\mathbf{u}\right)^{\frac{1}{2}} \\ &\text{DV}=\{\mathbf{u}, \mathrm{y}_{12}, \mathrm{y}_{21}\} \text{ and } \mathbf{u} = \{u_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4, \mathrm{u}_5\} \\ &\text{Subject to} \\ &\quad z_1 = F_{z1}(x_1, x_2, x_3, y_{21}) = 0 \quad \text{(DA1)} \\ &\quad y_{12} - F_{y12}(x_1, x_2, x_3, y_{21}) = 0 \quad \text{(DA1)} \\ &\quad y_{21} - F_{y21}(x_1, x_4, x_5, y_{12}) = 0 \quad \text{(DA2)}\end{aligned} \quad (32)$$

The MPPs obtained from both the proposed method (collaborative method) and the all-on-one method are listed in Table 1. Both methods generate almost identical solutions.

Table 1 MPP for Example 1 at Design Point 1

Method	$\mathbf{u}_{MPP}$=(u_1, u_2, u_3, u_4, u_5)
All-in-One Method	(2.3477, 1.9013, 0.9507, -0.0002, -0.0001)
Collaborative Method	(2.3477, 1.9014, 0.9507, 0.0, 0.0)

The reliability index β and the probability of failure p_f from three methods are shown in Table 2. The collaborative method and the all-in-one method produce the identical results. For the all-in-one reliability analysis method, the number of subsystem disciplinary analyses is 437 and the number of system-level multidisciplinary analyses is 56. On average, each system-level multidisciplinary analysis needs 7.8 disciplinary analyses. For the proposed collaborative reliability analysis method, the total number of disciplinary analyses is 152 and no system-level multidisciplinary analysis is needed. Therefore, the collaborative reliability analysis method is more efficient than the all-in-one reliability analysis method for this example. FORM (19) is used to calculate the probability of failure p_f. It is noted that the probabilities of failure p_f from both the all-in-one and the collaborative reliability methods are very close to the point estimate of p_f (in row 3) from Monte Carlo Simulation. In this case, the probability of failure p_f is very small and the Monte Carlo Simulation needs a large sample size to obtain an accurate solution. The interval estimate of p_f from Monte Carlo Simulation is also given in the footnote of the table.

The reliability analysis is also performed at design point 2 where the mean values $\mathbf{x}=\{x_1, x_2, x_3, x_4, x_5\}$ are $\boldsymbol{\mu}_x=\{2,5,2,5,2\}$, the limit-state function is considered with c=22, namely

$$F_{z1} = 22 - (x_1^2 + 2x_2 + x_3 + x_2 e^{-y_{21}}) \quad (33)$$

The results are listed in Tables 3 and 4. At design point 2, the collaborative reliability analysis method is again more efficient than the all-in-one reliability analysis method.

Table 2 Reliability Analysis Result for Example 1 at Point 1

Method	β	p_f	Number of DA[1]	Number of SA[2]
All-in-One Method	3.1671	7.6978×10^{-4}	437	56
Collaborative Method	3.1671	7.6978×10^{-4}	152	0
MCS[3]	3.1708	7.60×10^{-4}*	–	107

[1]DA – disciplinary analyses (subsystem)
[2]SA – system-level multidisciplinary analyses
[3]MCS – Monte Carlo Simulation
*The 95% confidence interval of p_f is (7.1467×10^{-4}, 8.0533×10^{-4})

Table 3 MPP for Example 1 at Design Point 2

Method	$\mathbf{u}_{MPP}$=(u_1, u_2, u_3, u_4, u_5)
All-in-One Method	(3.1328, 2.9819, 0.5962, -0.0001, 0.0003)
Collaborative Method	(3.1329, 2.9818, 0.5964, 0.0, 0.0)

Table 4 Reliability Analysis Result for Example 1 at Point 2

Method	β	p_f	Number of DA	Number of SA
All-in-One Method	4.3660	6.3274×10^{-6}	385	62
Collaborative Method	4.3660	6.3274×10^{-6}	136	0
MCS[1]	–	6.40×10^{-6}	–	107

[1]The 95% confidence interval of pf is (5.9840×10^{-4}, 6.8160×10^{-4})

Example 2 – Electronic Packaging Problem

The electronic packaging problem[12, 31, 32] is a benchmark multidisciplinary problem comprising the coupling between electronic and thermal subsystems. Component resistances (in electronic subsystem) are affected by operating temperatures in (thermal subsystem), while the temperatures depend on the resistances. The subsystem relationship is demonstrated in Fig. 7.

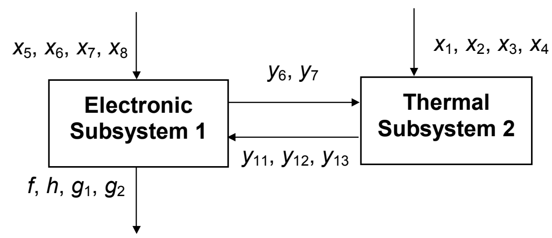

Figure 7. Information Flow - Electronic Packaging Problem

The system analysis consists of the coupled thermal and electrical analyses. The component temperatures calculated in the thermal analysis are needed in the electrical analysis in order to compute the power dissipation of each resistor. Likewise, the power dissipation of each component must be known in order for the thermal analysis to compute the temperatures.

There are eight random input variables x1– x8, five linking variables y_6, y_7, y_{11}, y_{12}, y_{13}, and four system outputs f, h, g_1, and g_2.

The sets of variables and functions in the two subsystems are shown as follows, where $\{\phi\}$ stands for an empty set.

Electronic Disciplinary Analysis:
Input variables: $\mathbf{x}_s=\{\phi\}$, $\mathbf{x}_1=\{x_5, x_6, x_7, x_8\}$
Linking variables: $\mathbf{y}_{21}=\{y_6, y_7\}$
Outputs: $\mathbf{z}_1=\{f, h, g_1, g_2\}$
Thermal Disciplinary Analysis:
Input variables: $\mathbf{x}_s=\{\phi\}$, $\mathbf{x}_2=\{x_1, x_2, x_3, x_4\}$
Linking variables: $\mathbf{y}_{12}=\{y_{11}, y_{12}, y_{13}\}$
Outputs: $\mathbf{z}_2=\{\phi\}$

Of the two subsystems, the thermal analysis is more complex, which requires a finite difference solution for the temperature distribution calculation. The remaining equations in the thermal subsystem are solved algebraically. All equations of the electrical system are solved algebraically.

g_1 and g_2 are considered as the limit state functions, which are the differences of the component temperature and the allowable temperature. We assume uncertainties are associated with the input variables x_i (i = 1, …, 8), described by normal distributions. The variation coefficient (the ratio of the standard deviation over the mean) of x_i is 0.1.

Table 5 Reliability Analysis Result for Example 1 for Limit State Function g_1

Method	β	p_f	Number of DA	Number of SA
All-in-One Method	2.7082	3.3825×10^{-3}	367	112
Collaborative Method	2.7127	3.3369×10^{-3}	111	0
MCS[1]	2.7144	3.320×10^{-3}	–	106

[1]The 95% confidence interval of pf is (3.2254×10^{-3}, 3.4146×10^{-3})

Table 6 Reliability Analysis Result for Example 1 for Limit State Function g_1

Method	β	p_f	Number of DA	Number of SA
All-in-One Method	3.0779	1.0×10^{-3}	531	164
Collaborative Method	3.0738	1.1×10^{-3}	169	0
MCS[1]	3.0357	1.15×10^{-3}	–	106

[1]The 95% confidence interval of pf is (1.0943×10^{-3}, 1.2057×10^{-3})

At the design $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} = \{0.08, 0.08, 0.055, 0.0275, 505.0, 0.0065, 505.0, 0.065\}$, the reliability index β and the probability of failure p_f from three methods for both limit states are shown in Tables 5 and 6 respectively. For limit state function g1, the collaborative method and the all-in-one method produce very close results. For the all-in-one reliability analysis method, the total number of subsystem disciplinary analyses is 367 and the number of system-level multidisciplinary analyses is 112, while the collaborative reliability analysis method uses only 111 subsystem disciplinary analyses and zero system-level multidisciplinary analysis. In this sense, the collaborative reliability analysis method is more efficient than the all-in-one reliability analysis method. FORM is used to calculate the probability of failure p_f. It is noted that the probabilities of failure p_f from both the all-in-one and the collaborative reliability methods are very close to the one from Monte Carlo Simulation. For limit state function g2, we have the similar conclusion. The collaborative method requires only 169 subsystem disciplinary analyses while 531 disciplinary analyses are used by the all-in-one method.

6. Concluding Remarks

In the traditional all-in-one reliability analysis method, the optimizer for locating the MPP repeatedly calls the limit state function which is evaluated at system-level wherein a number of individual disciplinary analyses are performed. Two nested loops are therefore involved in an all-in-one reliability analysis. The outer loop is the minimization problem for MPP search and the inner loop is the system-level analysis. The number of design variables of the minimization problem of an all-in-one reliability analysis is equal to the total number of random system input variables and random disciplinary input variables for all disciplines.

In contrast with the all-in-one reliability analysis, the collaborative reliability analysis method developed in this paper only employs a single optimization loop for MPP search. The interdisciplinary consistency (the system of simultaneous equations) is embedded in the optimization model for MPP search as equality constraints. In the process of searching the MPP, the interdisciplinary consistency is satisfied progressively. By this way, computations can be conducted concurrently at the individual disciplinary level. The design variables in the optimization for locating the MPP are random system input variables and random disciplinary input variables for all disciplines, as well as all the linking variables. Even though larger number of design variables (the difference is the total number of linking variables) may lead to more function evaluations in MPP search, the overall efficiency of the collaborative reliability analysis is generally superior to the all-in-one reliability analysis as demonstrated by the two examples in Section 5 due to the single loop procedure.

As for accuracy, both methods generally produce the same reliability estimations since both are based on the MPP concept for reliability assessment. It should be noted that besides the consideration of efficiency, depending on the existing computational framework for multidisciplinary analyses, one or the other method could be more favored. For instance, with the all-in-one reliability analysis, it is easier to integrate the existing reliability analysis methods/programs with an MDO framework where multidisciplinary analyses have been integrated at the system level. With the collaborative reliability analysis method, the optimization problem for MPP search with interdisciplinary consistency needs to be customized by a designer. However, the collaborative reliability analysis method could be more favored under a distributed computing environment. It should also be noted that both methods in principle are gradient based and therefore the computational effort is approximately proportional to the number of random input variables (as well as linking variables for the collaborative method). With extremely high problem dimensions, the Monte Carlo Simulation can be considered as an alternative[31].

The proposed method is demonstrated in this paper only for the purpose of reliability analysis under the MDO framework. When we perform MDO under uncertainty, for example, robust MDO and reliability-based MDO, the techniques discussed herein can be utilized to evaluate any probabilistic objectives and probabilistic constraints. For MDO under uncertainty, the reliability analysis is called repeatedly by the MDO optimizer. In other words, the reliability analysis loop will be embedded in the optimization loop of the MDO. If the all-on-one reliability analysis method is adopted, the procedure of an MDO becomes a triple-loop. As a result, the computation will be prohibitively expensive. However, if we use the proposed collaborative reliability analysis method, only two-loop procedure is needed and therefore the computational burden is mitigated.

No matter which reliability analysis method is employed, evaluating probabilistic constraint directly under MDO optimizer always introduces nested loops. As a part of the future work, we plan to develop more efficient strategies and methods, ideally, single-loop strategy, to suit the features of probabilistic design under the MDO environment.

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