Probabilistic Sensitivity Analysis Methods for Design under Uncertainty

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Sensitivity analysis (SA) is an important procedure in engineering design to obtain valuable information about the model behavior to guide a design process. For design under uncertainty, *probabilistic sensitivity analysis (PSA)* methods have been developed to provide insight into the probabilistic behavior of a model. In this paper, the goals of PSA at different design stages are investigated. In the prior-design stage, PSA can be utilized to identify those probabilistically non-significant variables and reduce the dimension of a random design space. It can reduce the computational cost associated with uncertainty assessment without much sacrifice on the optimum solution. For post-design analysis, probabilistic sensitivity analysis can be used to identify where to spend design resources for the largest potential improvement of a performance. Based on the interested distribution range of a random response, the PSA methods can be categorized into two types: the global response probabilistic sensitivity analysis (GRPSA) and the regional response probabilistic sensitivity analysis (RRPSA). Four widely-used PSA methods: Sobol' indices, Wu's sensitivity coefficients, the MPP based sensitivity coefficients, and the Kullback-Leibler entropy based method are selected for comparison. The merits behind each method are reviewed in details. Their advantages, limitations, and applicability are investigated. Their effectiveness and applicability under different design scenarios are compared in two numerical examples and two engineering design problems.

Key words: probabilistic sensitivity analysis, robust design, reliability-based design, sensitivity coefficient, main effect, total effect, variance-based methods, Kullback-Leibler entropy

I. Introduction

SENSITIVITY analysis (SA) has been widely applied in engineering design to explore the model response behavior, to evaluate the accuracy of a model, to test the validity of the assumptions made, etc. In deterministic design, sensitivity analysis is used to find the *rate of change* in a model output due to changes in the model inputs. That is usually performed by varying input variables *one at a time* near a given central point, which involves partial derivatives and often called *local sensitivity analysis*.

It has been widely acknowledged that uncertainty is inevitable in a product development process. Robust design[1,2] and reliability-based design[3,4] are two widely used probabilistic design methods that have gained wide attentions to ensure the quality of a product under uncertainty. Robust design is used to minimize the effect of variations in controllable and/or uncontrollable factors without eliminating the sources of variations, while the reliability-based design has been widely applied to ensure that a system performance meets the pre-specified target with a required probability level. Though it is important to seek the optimal solution in design under uncertainty, sensitivity analysis is also important for designers to gain insights about the complex model behavior and make informed decisions regarding where to spend the engineering effort to reduce the variability of a system.

When uncertainty is considered, sensitivity analysis has different meanings. We assume that the uncertainty in a design performance is described probabilistically by its mean (μ), variance (σ^2), the probability density function (PDF), or the cumulative distribution function (CDF), etc. Correspondingly, the sensitivity analysis under uncertainty needs to be performed on the probabilistic characteristics of a model response with respect to the

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probabilistic characteristics of model inputs. In general, *the* ***probabilistic sensitivity analysis (PSA)*** *is a study to quantify the impact of uncertainties in random variables on the uncertainty in the model output*. Results from PSA have been used to assist engineering design from various aspects, such as to help reduce the dimension of a design problem by identifying the probabilistically insignificant factors; to check the validity of a model structure and the assumptions made on the probability distributions of random inputs; to obtain insights about the design space and the probabilistic behavior of a model response; and to investigate potential improvement on the probabilistic response by reducing the uncertainty in random inputs[5]. Various probabilistic sensitivity analysis methods exist in the literature, however, a good understanding of their usages and the relative merits of different methods is certainly lacking. In this paper, we review the representative PSA methods including our newly developed relative entropy based PSA method. We then study the relative merits of four major categories of PSA methods.

A popular category of the existing PSA techniques belongs to the so-called variance-based methods, including the Fourier Amplitude Sensitivity Test (FAST)[6,7], correlation ratios[8] or importance measures[9], and Sobol' indices[10,11], etc. Variance-based methods are derived from the decomposition of the total variance of a model response to different variation sources and their combinations. FAST provides a way to evaluate a variance by converting a multi-dimensional integral to a one-dimensional integral. Sobol' method for variance estimation is based on an ANOVA-like decomposition of a function with an increasing dimensionality. Correlation ratio, also referred to as importance measures, or their variants are based on the evaluation of variance of a conditional expectation. Obviously, the variance-based methods can be directly applied to PSA in robust design as they matches with the objective of minimizing the response variance in robust design.

Another widely used category of PSA techniques is to investigate the rate of change in a probabilistic characteristic of a response Y due to the changes in the probabilistic characteristics of a random input X_i, such as $\partial\mu_Y/\partial\mu_{X_i}$ and $\partial\mu_Y/\partial\sigma_{X_i}$. In particular, for reliability-based design, the sensitivity of the failure probability (P_f) is of interest, for example, $\partial P_f/\partial\mu_{X_i}$ and $\partial P_f/\partial\sigma_{X_i}$. Wu[12] proposed a normalized sensitivity coefficient of a failure probability with respect to a random variable as an expectation of the partial derivatives of the performance PDF, evaluated over the failure region. Mavris, et al.[13] extended the above coefficients to the sensitivity of any probabilistic characteristics of a performance. Based on the Kuhn-Tucker condition satisfaction at the most probable point (MPP) of failure, another sensitivity measure related to reliability is defined as the gradient of a limit state function at the MPP in the standard normal space, normalized by the reliability index[14]. The reliability sensitivity based on MPP can be interpreted as the decomposition of the reliability index onto each dimension of a random space, representing the contribution of each random variable to the reliability.

In our recent development[15], we proposed relative entropy based measures of probabilistic sensitivity and demonstrated their applications in various design scenarios. Entropy, as a measure of uncertainty in a random variable, has been used as importance measures in decision making[16,17]. Kullback-Leibler (K-L) entropy, or called relative entropy, measures the divergence from one probability distribution to another[18]. Although not a metric itself, K-L entropy shares many properties of a metric. Based on the concept of "omission sensitivity"[19], K-L entropy is able to measure the change of a performance distribution by removing all uncertainty in a random input, i.e., replacing it with a deterministic value, say, its nominal value. The larger the change in a performance distribution due to fixing a random input, the more important that random factor is. By comparing such changes due to the uncertainty elimination in different random inputs, K-L entropy can capture the relative importance of random inputs.

Although sensitivity analysis under uncertainty has gained a lot of attention, we find that there is no good understanding of the relative merits of the different methods and the use of the different terms are sometimes confusing. In this paper, we focus on the comparison of four representative PSA methods: Sobol' indices (as an example of the variance-based method), Wu's sensitivity coefficients, reliability sensitivity based on MPP, and our proposed relative entropy based PSA method. By investigating the metrics behind different PSA methods, we discuss their advantages, limitations, and applicability. Using examples, we demonstrate their applications under different design scenarios and at different design stages.

II. Goals and Application Scenarios of PSA

The choice of a suitable PSA approach largely depends on the purpose of conducting sensitivity analysis, for example, whether the goal is to reduce the dimension of a design space or to investigate the potential improvement on a performance behavior. This raises the question: *what information do we expect to draw from the probabilistic sensitivity analysis?* The answer to this question largely depends on the design formulations as well as the design stages in which PSA is performed.

A. Goals of PSA

Traditionally, sensitivity analysis is performed in the post-design stage after a design solution is identified. There is also a great need for designers or modelers to conduct PSA in the prior-design stage to gain valuable information about the model and its probabilistic behavior. This is especially important for models with high dimensions (i.e., a large number of random inputs and/or a large number of performances) as well as for models with high nonlinearity such that the relationships between inputs and outputs are not obvious. Based on whether the PSA is conducted across the whole range of a design space or at a particular design solution, we categorize PSA into those for the prior-design and those for the post-design stage.

Prior-Design

The goal of prior-design PSA is aimed to answer the following question:

Which variable(s) could be safely eliminated without bringing much influence on the uncertainty in the response?

Because of the computational efforts associated with uncertainty propagation, there is always a need to reduce the size of a probabilistic design problem by eliminating insignificant (controllable) design variables—either deterministic or random variables that engineers choose to control their nominal setting to "optimal" values—and (uncontrollable) random (noise) parameters—random variables that engineers choose not to control. Based on the importance ranking of all variables, unimportant design variables and noise parameters could be treated as deterministic variables and fixed as constants. When applying the PSA across the whole design space, both the deterministic and random (controllable) design variables are considered to be uniformly distributed over their entire range, while the random noise parameters follow the pre-specified distributions.

Post-Design

Once a design solution is identified through optimization, the focus of the PSA in the post-design stage is to answer the following question:

Which random uncertainties should be further controlled (eliminated) to gain the largest improvement on the probabilistic performance of a response?

An example of the above case is "tolerance design" where manufacturing precision is improved to reduce or eliminate variations of design variables. The post-design PSA is applied to prioritize available resource for variance reduction. Conversely, tolerance requirements can be relaxed for noise factors with negligible effects to output variation to reduce the computational cost. Although the post-design PSA could not tell directly whether it is worthwhile to spend extra design resources, it does indicate the effective way to spend resources if there is a need to further improve a performance behavior, such as reducing performance variance. In particular, if a performance in the reliability-based design could not meet a specific probability level, PSA is used at this stage to decide which variability should be reduced further.

B. Probabilistic Design Scenarios

Probabilistic Sensitivity Analysis (PSA) studies the impact of uncertainties in design variables and noise parameters on the probabilistic characteristics of a design performance. The word *impact* has different meanings under different design scenarios. For robust design, the goal of PSA is to identify those random variables which contribute the most to the variance of a response *y*; in other words, if reducing the uncertainties in these random variables, the variance of a response could be reduced at the most. For reliability-based design, it means the contribution of reducing different sources of uncertainty to the improvement of the probability of a design constraint satisfaction. In other words, the objective is to identify those random inputs which have the most influence on the probability of meeting a pre-specified target. This is particularly important if a target could not be satisfied with a required probability level. In such a case, PSA indicates where to spend efforts, if available, to gain as large reliability improvement as possible.

Considering the situation above, existing PSA methods can be categorized based on the performance distribution range of interest, as shown in Figure 1. In robust design, because the design objective is to reduce the variance of a response, the full range distribution of a response, i.e., $[-\infty,+\infty]$ is of interest. On the other hand, in reliability-based design, the interest is on assessing the impact on preventing the failure in a local region, say $[-\infty, y^{\alpha}]$ or $[y^{\alpha}, y^{\beta}]$. Accordingly, we define two types of PSA methods:

- **Global response probabilistic sensitivity analysis (GRPSA)** — PSA for the case that the interest is among the entire distribution of a response;

- **Regional response probabilistic sensitivity analysis (RRPSA)** — PSA for the case that the interest is among a partial range of a response distribution, either at the tail of a distribution for reliability-based design (e.g., [-∞, y^{α}]), or within any distribution range [y^{α}, y^{β}] in a general sense.

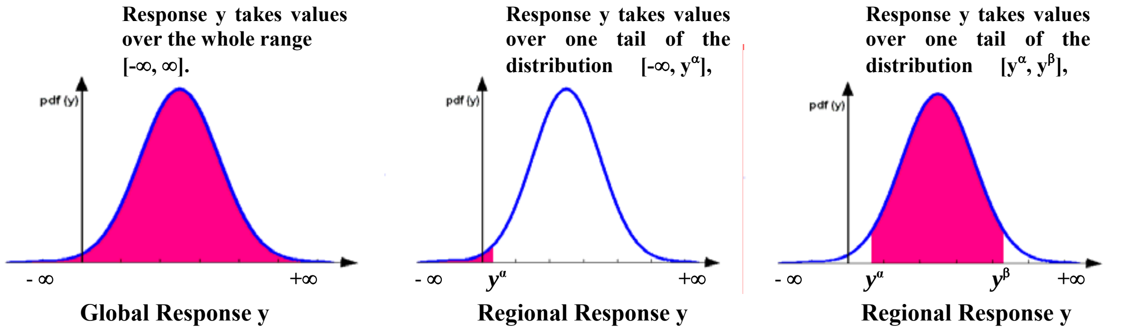

Figure 1. Global vs. Regional Response Range

III. Comparison of Four PSA Methods

In the context of the PSA goals and design scenarios discussed in Section 2, we hereby investigate the relative merits of four representative PSA methods and discuss their advantages, applicability, and limitations. With the comparison, we aim to develop guidelines for selecting an applicable PSA in different design situations. The computational resource required for each method is also discussed. In this investigation, we only consider variables with continuous distributions. The method can be directly extended to discrete random variable cases.

A. Variance-Based Methods—Sobol' Indices

The variance-based approaches for sensitivity analysis are based on the decomposition of the variance of a response (Y) to its variation sources:

$$V = \sum_i V_i + \sum_{i<j} V_{ij} + \ldots + V_{1,2,3,\ldots,n} \tag{1}$$

The first order terms V_i represent the partial variance in the response due to the individual effect of a random variable X_i, while the higher order terms show the interaction effects between two or more random variables. The above decomposition put forward two important concepts: the main effect and the total effect. The former refers to the effect of a term associated with only one random variable. The latter includes both the individual effect of a random variable as well as its interaction with other random variables. The *main effect index* (MSI) of a random variable X_i is obtained by the normalization of the main-effect variance over the total variance in Y as shown in Equation 2. Equation 3 gives the sensitivity index for the interaction between two random variables, X_i and X_j. A general sensitivity index is given in Equation 4.

$$S_i = V_i / V, \quad 1 \le i \le m. \tag{2}$$

$$S_{ij} = V_{ij} / V, \quad 1 \le i < j \le m. \tag{3}$$

$$S_{i,\, i+1,\ldots,m} = V_{i,\, i+1,\ldots,m} / V \tag{4}$$

When there is a significant interaction among random variables, evaluation of main effects only is not enough. In such situation, the *total effect* of a random variable X_i, which includes its main effect and all the interaction effects involving X_i, is required to accurately describe its contribution. By partitioning the whole set of variables into a subset of interest and its complementary[11], i.e., X_i and $\mathbf{X}_{\sim i}$, where the latter is the subset of all variable excluding X_i, the *total effect index* (TSI) is given by

$$S_{Ti} = 1 - S_{\sim i} \tag{5}$$

where, $S_{\sim i} = S_{1,\ldots,i-1,i+1,\ldots,m}$ is the index for the combined effect of all random variables except X_i.

There are many approaches to obtain the above sensitivity indices. The impact of a random input can be evaluated through the reduction in the variance of a response (Y) contributed by fixing that random variable. The ratio $V_i / V = V[E(Y \mid X_i)]/V$ is known as the correlation ratio or called importance measures[7-9]. FAST method uses the Fourier analysis to avoid the evaluation of the multi-dimensional integration. However, the original correlation ratios and the FAST methods are for evaluating the main effect only. Researchers have extended their applications to the total effect assessment[7]. However, those approaches are not able to obtain the interaction effects between random inputs. The computational issues about the correlation ratios and the complication of numerical implementation behind the FAST method have limited the uses of these methods.

Sobol'[10,11] proposed a more practical approach which uses an unique decomposition of a function into summands with increasing dimensions as

$$f(\mathbf{x}) = f_0 + \sum_{i=1}^{m} f_i(x_i) + \sum_{i=1}^{m}\sum_{j=i+1}^{m} f_{ij}(x_i, x_j) + \ldots + f_{1,\ldots,m}(x_1,\ldots,x_m), \tag{6}$$

where **x** is a vector of m variables. f_0 is a constant. f_i is a function of X_i only. f_{ij} is a function of x_i and x_j only, and so on. Then the variance and partial variance terms in Equation 1 become:

$$V = \int f^2(\mathbf{x}) p(\mathbf{x}) d\mathbf{x} - f_0^2 \tag{7}$$

$$V_{i,\ldots,m} = \int f_{i,\ldots,m}^2(x_i,\ldots,x_m) p(x_i,\ldots,x_m) dx_i \ldots dx_m \,, \tag{8}$$

where $p(\mathbf{x})$ is the joint PDF of random variables, **X**. Equations 7 and 8 can be evaluated by Monte Carlo methods to obtain the main effects and total effects defined in Equations 2-5. The interaction effects can also be obtained but requiring the multidimensional integration. Sobol'[11] developed Monte Carlo estimates for the total effects, which requires the same computational expense as for the main effects. In this paper, we choose Sobol' method as a representative variance-based method as a comparison benchmark for other methods.

All variance-based methods are based on the evaluation of either the conditional or partial variance of an output. Variance-based methods are GRPSA approaches as the variance is calculated over the entire range of a performance distribution. This category of methods can be utilized when the variance of a response is of interest, such as in robust design. They can be applied at both the prior-design and the post-design stages. In both stages, variance-based approaches can generate an importance ranking of all random variables. For the prior-design situation, the ranking can help designers identify those random variables with little potential impact on the response variability. Thereafter, the dimension of the probabilistic design space can be reduced as well as the computational cost. For the post-design analysis, the ranking provides valuable information on where to spend additional resources to further control the source of variations.

The major limitations of variance-based methods are: (1) They assume that the second moment (performance variance) is sufficient to describe the uncertainties encountered. This type of methods may lose their accuracy when the variance is not a good measure of the distribution dispersion such as the case where the response distribution has high skewness and kurtosis. (2) They cannot be applied for studying the effect of a random variable on a performance over a partial region of distribution, such as the failure region. They are not applicable for RRPSA.

B. Probability Sensitivity Coefficients

For those probabilistic design problems in which the probability of a response violating or meeting a pre-selected target is of interest, a heuristic probability-based sensitivity measure is defined as the rate of change in a probability (P) due to changes in a statistical parameter (θ_i) of a random input, as $\partial P/\partial \theta_i$. Although written in the form of a partial derivative, it needs to be evaluated over the range on which the probability P is defined. It is usually impossible to get a close solution for the probability sensitivity coefficients. One way is to calculate $\partial P/\partial \theta_i$ numerically using the concept of finite difference as[19]:

$$S_{\theta_i} = \frac{(P + \Delta P) - P}{\Delta \theta_i} \tag{9}$$

where, θ_i is a uncertainty measure, which is usually taken as one of the parameters which describe the distribution of a random variable X_i, such as the mean or the variance. Obviously, the accuracy of S_{θ_i} may be influenced by the choice of $\Delta\theta_i$ value.

Expanding the analytical equation of $\partial P/\partial \theta_i$, Wu[12,20] derived the so-called CDF-based sensitivity coefficients for the probability of failure (P_f) as:

$$S_{\theta_i} = \frac{\partial P_f / P_f}{\partial \theta_i / \theta_i} = \int_\Omega \ldots \int \frac{\theta_i}{P_f} \cdot \frac{\partial p(\mathbf{x})}{p(\mathbf{x}) \partial \theta_i} p(\mathbf{x}) d\mathbf{x} = \int_\Omega \ldots \int \frac{\theta_i}{p(\mathbf{x})} \cdot \frac{\partial p(\mathbf{x})}{\partial \theta_i} \left[\frac{p(\mathbf{x})}{P_f} \right] d\mathbf{x} = E\left[\frac{\theta_i \, \partial p(\mathbf{x})}{p(\mathbf{x}) \partial \theta_i} \right]_\Omega \tag{10}$$

where, $p(\mathbf{x})$ is the joint probability density function of all random variables for a failure mode. Ω denotes the failure region. Equation 10 is the expectation of the normalized partial derivative of the joint PDF with respect to a distribution parameter over the failure region. Different from Equation 9, Wu's sensitivity coefficients are the average impact of θ_i on the probability of failure. They are usually evaluated by sampling methods. P_f can be estimated by empirical CDF. After selecting a specific target or a required probability level, the failure range of a performance, say Y, is identified as well as the corresponding values of **x**. Then S_{θ_i} is computed by taking the average of $\frac{\theta_i \partial p(\mathbf{x})}{p(\mathbf{x})\partial\theta_i}$ based on samples within the failure region. The values of S_{θ_i} could be positive, zero, or negative. For comparison of importance, the absolute values should be used instead.

Equation 10 can be further simplified if all random variables are transformed to the standard normal space, as shown in Equations 11 and 12.

$$S_{\mu_i} = \frac{\partial P_f / P_f}{\partial \mu_i / \sigma_i} = \int \ldots \int_{\Omega} \frac{\sigma_i}{P_f} \cdot \frac{\partial \phi(\mathbf{u})}{\partial \mu_i} d\mathbf{u} = \int \ldots \int_{\Omega} u_i \left[\frac{\phi(\mathbf{u})}{P_f}\right] d\mathbf{u} = E[u_i]_{\Omega} \tag{11}$$

$$S_{\sigma_i} = \frac{\partial P_f / P_f}{\partial \sigma_i / \sigma_i} = \int \ldots \int_{\Omega} \frac{\sigma_i}{P_f} \cdot \frac{\partial \phi(\mathbf{u})}{\partial \sigma_i} d\mathbf{u} = \int \ldots \int_{\Omega} \left(u_i^2 - 1\right) \left[\frac{\phi(\mathbf{u})}{P_f}\right] d\mathbf{u} = E\left[u_i^2\right]_{\Omega} - 1\,, \tag{12}$$

where **u** is a vector of standard normal random variables transformed from **X**. If **X** follows independent normal distributions, then the transformation is simply $u_i = \frac{X_i - \mu_i}{\sigma_i}$. ϕ is the joint PDF of **u**. Although derived for the failure probability, Wu's sensitivity coefficients can be extended to any partial region in the design space corresponding to a probability.

Another probability sensitivity coefficient is related to the reliability-based design applications, based on the concept of the most probable point (MPP) of failure, as illustrated in Figure 2. MPP, defined in the standard normal space, is the point on the limit state $y(\mathbf{X})=0$ that contributes the most to the probability $P\{y(\mathbf{X})=0\}$. The reliability index $\beta=\Phi^{-1}(P\{y(\mathbf{X})=0\})$ is the shortest distance from the origin to the limit state in the standard normal space. By projecting the β vector onto each dimension of the random design space, such as u_1 and u_2, the components along each dimension normalized by β provide sensitivity indicators of the reliability with respect to random variables, as shown in Equation 13[14].

$$S_i = \left.\frac{\left(\frac{\partial y}{\partial x_i}\frac{\phi(u_i)}{h(x_i)}\right)^2}{\sum_{j=1}^{n}\left(\frac{\partial y}{\partial x_j}\frac{\phi(u_j)}{h(x_j)}\right)^2}\right|_{MPP} = \frac{\left(u_i^{MPP}\right)^2}{\beta^2} \tag{13}$$

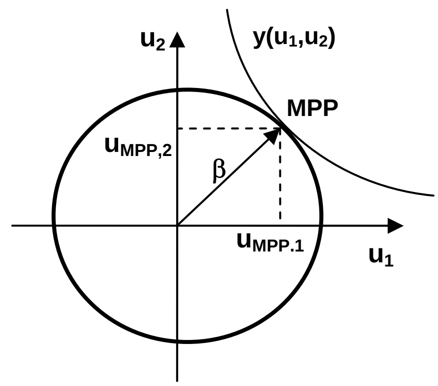

Figure 2. Illustration of the MPP-based sensitivity measures

where, y is a random performance, $\phi(\cdot)$ is the PDF of the standard normal distribution, $h(\cdot)$ is the PDF of a random variable, X_i. u_i is the standard normal random variable transformed from X_i. β is the reliability index. It should be noted that $\sum_{i=1}^{n} S_i = 1$. Moreover, S_i is the directional cosine in the gradient of the limit state at the MPP. For a reliability-based design utilizing the MPP information for reliability assessment, Equation 13 does not require additional computational efforts. The sensitivity factors S_i become by-products.

As any probability is pertained to a partial range of a response distribution, all the probability sensitivity coefficients discussed above are RRPSA methods, not applicable to studying the impact on the whole distribution of a performance.

C. Kullback-Leibler Entropy Based PSA Method

The variance-based methods are applicable to GRPSA only, while the probability sensitivity coefficients are limited to RRPSA. To overcome the limitations of existing methods, we proposed a unified PSA method based on the concept of Kullback-Leibler (K-L) entropy[18]. The K-L entropy is defined as

$$D_{KL}(p_1 \mid p_0) = \int p_1(y) \cdot \log \frac{p_1(y)}{p_0(y)} dy = E_{p_1}\left[\log \frac{p_1(y)}{p_0(y)}\right] \tag{14}$$

Although not a true metric, the relative entropy shares many properties of a metric, such as non-negativity, additive property, and convexity. The relative entropy is a measure of the averaged lack of overlapping between two PDF curves over a region specified by the integration limits. It is assumed that a random response Y = h(**X**) has a PDF of p_0, where **X** denotes a vector of random inputs. When fixing a random input X_i at its mean value, i.e., eliminating all of its uncertainty, the PDF of Y changes to p_1. Therefore, the relative entropy can evaluate the total effect of X_i on the distribution of Y by measuring the difference between two distributions: p_0 and p_1. The combined effects of the complementary of $\mathbf{X}_i$, $D_{KL\sim x_i}$, can be obtained by fixing all random variables except X_i and studying the change of the response PDF. The main effect of $\mathbf{X}_i$ is the reverse of $D_{KL\sim x_i}$. By specifying the integration limits for the K-L entropy computation, the method can be applied both globally and regionally.

For GRPSA, a K-L entropy based method measures the total and main effect indices of X_i as follows:

$$D_{KLx_i}(p_1 \mid p_0) = \int_{-\infty}^{\infty} p_1(y(x_1,...,\bar{x}_i,...,x_n)) \cdot \log \frac{p_1(y(x_1,...,\bar{x}_i,...,x_n))}{p_0(y(x_1,...,x_i,...,x_n))} dy \tag{15}$$

$$D_{KL\sim x_i}(p_1 \mid p_0) = \int_{-\infty}^{\infty} p_1(y(\bar{x}_1,...,x_i,...,\bar{x}_n)) \cdot \log \frac{p_1(y(\bar{x}_1,...,x_i,...,\bar{x}_n))}{p_0(y(x_1,...,x_i,...,x_n))} dy \tag{16}$$

where, $\bar{x}_i$ means fixing X_i at a value, usually chosen at its mean, μ_{x_i}. The larger the $D_{KLx_i}(p_1 \mid p_0)$, the more important the X_i is. The smaller the $D_{KL\sim x_i}(p_1 \mid p_0)$, the more important the main effect of X_i is. It should be noted that $D_{KL\sim x_i}(p_1 \mid p_0)$ itself is not the main effect, but it can be used to interpret the main effect.

With simple adjustments in the formulae, the proposed K-L method can also be used for RRPSA over a partial range of interest [y_L, y_U]. The total and main effect indices of X_i are defined in Equations 17 and 18, respectively.

$$D_{KLx_i}(p_1 \mid p_0) = \int_{y_L}^{y_U} p_0(y(x_1,...,\bar{x}_i,...,x_n)) \cdot \left|\log \frac{p_1(y(x_1,...,\bar{x}_i,...,x_n))}{p_0(y(x_1,...,x_i,...,x_n))}\right| dy \tag{17}$$

$$D_{KL\sim x_i}(p_1 \mid p_0) = \int_{y_L}^{y_U} p_0(y(\bar{x}_1,...,x_i,...,\bar{x}_n)) \cdot \left|\log \frac{p_1(y(\bar{x}_1,...,x_i,...,\bar{x}_n))}{p_0(y(x_1,...,x_i,...,x_n))}\right| dy \ . \tag{18}$$

For reliability-based design, the integration range in above equations will be changed to the ranges that correspond to the tails of a distribution. To ensure the validity of the K-L based approach for RRPSA, the absolute value of the log-likelihood is used. Also, p_0 is used as a weighting factor applied in front of the log-likelihood, instead of p_1. Over a partial region [y_L, y_U], instead of evaluating the averaged lack of overlapping between p_0 and p_1, the absolute divergence over that region is measured.

Over the entire range of a response distribution, the effect of a random variable is measured by its impact on the whole distribution of that response. Over a specific region, the effect of a random variable is indicated by its impact on the distribution of the response within that range. The whole distribution captures the complete uncertainty information beyond the mean and variance. This higher order moment differentiation is necessary because two distribution curves with the same variance could still have different distribution shapes. Obviously, K-L based method gives a more complete measure of the effect of a random variable than variance-based measure. It should be noted that the K-L methods can only give a relative importance ranking of random variables. The absolute values of the K-L measures themselves are hard to interpret. Unlike a true metric, there is not yet any method to normalize the K-L values obtained from Equations 15-18. The PDFs in the integral are usually obtained by sampling-based estimations. The integral can be computed by numerical methods. The summary of the comparison of four methods studied, as well as the computational issue, is investigated in the following part.

D. Comparison of four PSA methods

Based on the introduction of the four representative PSA approaches (Sobol's indices, Wu's sensitivity coefficients, MPP-based sensitivity factors, and K-L based sensitivity methods) in the proceeding subsection, we summarize here the applicability of these methods and discuss the related computational issues.

As one of the most widely-used variance-based method, Sobol' indices can provide the main effect, interaction effects, and the total effect of a (group of) random variable(s). The total effects can be obtained at the same computational cost as the main effects. With two sets of Monte Carlo sampling, the main and the total effects of all random variables can be obtained. Sobol' method can be directly applied to robust design for both prior-design and post-design sensitivity analysis. The disadvantage of the approach mainly lies in two aspects: (1) Sobol' method is a GRPSA method, which can only measure the *global* variability of the output over the entire range of the input variables. It is not applicable to any partial region of random distribution. (2) In a situation where variance is not sufficient to describe the uncertainty in a response, e.g., when a response distribution is highly skewed and heavily tailed, Sobol' indices may no longer be good PSA indicators.

Contrary to the variance-based PSA methods, Wu's sensitivity coefficients and the MPP-based sensitivity coefficients are only applicable for the RRPSA over the failure region to assess the impact on the probability of failure or reliability. They are more applicable at the post-design stage for identifying the critical random variable at a particular design setting. Wu's coefficients provide the first-order (linear) effects of a distribution parameter, such as mean or variance of a random variable on the probability of failure averaged over the failure region. It is very difficult to compute Wu's sensitivity coefficients analytically. The evaluation could be computationally expensive because of the multi-dimensional integration. If all random variables can be transformed to the standard normal space based on the CDF information, those coefficients could be much simplified and efficiently evaluated by importance sampling. The major limitations of Wu's approach are: (1) It could not be applied globally, even though the method can be repeated at different percentiles (probability) to gain some idea about the changes in sensitivity over the whole distribution range. (2) It is mainly developed for post-design applications. (3) The approach can only provide sensitivity information with respect to a single statistic, such as mean or variance. It is not able to provide the effect of all uncertainty in a random variable.

On the other hand, the MPP-based sensitivity coefficient investigates the impact of all uncertainty in a random variable on the reliability. In reliability-based design, if the MPP information is available from the reliability assessment in the design procedure, there is no additional computation cost for sensitivity analysis. The MPP-based sensitivity method shares the first two limitations as the Wu's approach. It can also be only used over a partial region which corresponds to a specific probability, but not the entire range of a response. The only way to get some insight about that sensitivity over the whole range is to repeat the RRPSA at multiple probability levels.

The K-L entropy based method is the only approach that can be applied both globally (Equations 15 and 16) and regionally (Equations 17 and 18). It can also be utilized both at prior design for the screening purpose and at the post design for further uncertainty reduction. Based on the divergence between two distribution curves, the complete uncertainty beyond variance is captured. Therefore, the K-L entropy based method provides a more informative sensitivity measure than those based on finite moments. The major limitation of the K-L based methods is the difficulty of interpreting the absolute values of the sensitivity results. Unlike the above three methods that can generate normalized values of sensitivity indices, the sensitivity information from the K-L entropy based approaches can only give a relative importance ranking of random variables, but not normalized results.

The computational cost of the proposed K-L entropy measures is mainly spent on the estimation of two PDFs: one before and one after uncertainty reduction in random inputs. For low-cost model, Monte Carlo simulations can be employed. For high-fidelity and expensive simulation models, the PDF estimation via Monte Carlo simulation is unaffordable. To overcome the computational barrier, one approach is to use the metamodeling techniques[21] to build surrogate models as approximations of high-fidelity models. Sampling techniques are then applied to the easy-to-compute metamodels. Based on samples, the PDF of a random response could be obtained by kernel density estimation (KDE)[22]. As an alternative to the sampling-based methods, the PDF information could be obtained by the most probable point-based uncertainty analysis (MPPUA) method[23] or the First Order Saddlepoint Approximation[24].

The applicability of the four PSA methods to global and regional PSAs are summarized in Table 1 for both prior and post design stages.

Table 1. Comparison and Applicability of Four PSA Methods

		Sobol' Indices	Wu's Sensitivity Coefficients	MPP-Based Sensitivity Coefficients	K-L Entropy Based PSA Method
Global	Prior Design	Yes	No	No	Yes
	Post Design	Yes	No	No	Yes
Regional	Prior Design	No	Yes	Yes	Yes
	Post Design	No	Yes	Yes	Yes
Computational Cost		Two Monte Carlo Sampling	Importance Sampling	MPP (no additional cost if using MPP-based reliability analysis)	Sampling-based methods like KDE for PDFs; numerical methods for integration

IV. Examples

In this section, the effectiveness and applicability of the four representative PSA methods discussed above are compared using both numerical and engineering design examples. The first numerical example with a linear model is chosen to demonstrate the use of Sobol' method and the K-L based method for GRPSA and to explain how to interpret the results from global sensitivity analysis. The second numerical example is selected to show a situation where the Sobol' method and the K-L based method give different importance ranking for GRPSA. The two engineering design problems are chosen to show the use of PSA methods under different design scenarios. The K-L based methods, Wu's sensitivity coefficients, and the MPP-based sensitivity factors are compared for the reliability-based design of a speed reducer as an example for the RRPSA. Using the same example, the Sobol' method and the relative entropy-based method are compared for prior-design GRPSA, applied for an integrated robust and reliability design formulation. The effectiveness of these methods is verified either graphically or by confirming the probabilistic design results.

A. Numerical Examples

1. GRPSA—Linear Model

A simple linear model $y = x_1 + 2x_2 + 3x_3$ is considered with three independent random variables X_1, X_2, and X_3, all following the normal distribution $N(\mu, \sigma^2)$ with $\mu = 2$ and $\sigma^2 = 0.04$. For GRPSA, using the Sobol' method, the variance of the response Y can be decomposed as:

$$V_y = V_{x_1} + V_{x_2} + V_{x_3} = \sigma^2 + 4\sigma^2 + 9\sigma^2 = 14\sigma^2 \tag{19}$$

Because there is no interaction between any two variables, the main effect of a random input is also its total effect, i.e.,

$$S_i = S_{Ti} = V_{x_i} / V \tag{20}$$

Based on Equation 20, we get

$$S_1 : S_2 : S_3 = S_{T_2} : S_{T_2} : S_{T_3} = 1/14 : 4/14 : 9/14 = 1 : 4 : 9 \tag{21}$$

Results in Equation 21 indicate that X_3 is the most important factor, which has the largest impact on the variance of Y.

Utilizing the K-L entropy-based method, the relative importance ranking can be obtained based on the main and the total effects. The analytical approach is used to evaluate the total effects of three variables based on Equation 15. Due to the linear relationship, Y follows a normal distribution $N(\mu_y, \sigma_y^2)$ with μ_y=12, σ_y^2= 0.56 and the probability density function $p_0(y) = \frac{1}{0.56\sqrt{2\pi}} \exp\left[-\frac{(y-12)^2}{2(0.56)^2}\right]$. When X_1 is fixed at its mean value, Y still follows a normal distribution but with a difference variance, i.e., Y(μ_{X1}, X_2, X_3) following N(12, 0.52) with the PDF as $p_1(y) = \frac{1}{0.52\sqrt{2\pi}} \exp\left[-\frac{(y-12)^2}{2(0.52)^2}\right]$. When X_2 is fixed at its mean value, Y(X_1, μ_{X2}, X_3) follows N(12, 0.4) with $p_1(y) = \frac{1}{0.4\sqrt{2\pi}} \exp\left[-\frac{(y-12)^2}{2(0.4)^2}\right]$. Similarly, when X_3 is fixed at its mean, Y(X_1, X_2, μ_{X3}) follows N(12, 0.2) and $p_1(y) = \frac{1}{0.2\sqrt{2\pi}} \exp\left[-\frac{(y-12)^2}{2(0.2)^2}\right]$. The total effect of a variable is computed by measuring the difference between p_1 and p_0 when eliminating the entire uncertainty in a random variable. The larger the divergence, the larger the total effect is. The indices of the total effects by the K-L method are obtained as

$$KL_{x_1} : KL_{x_2} : KL_{x_3} = 0.0013 : 0.0254 : 0.1934 = 1 : 18.9438 : 144.3465 \tag{22}$$

It is observed from both Equation 22 and Figure 3 that X_3 is the most critical random variable in terms of its impact on the uncertainty in Y. X_2 is the second important and X_1 is the most insignificant factor.

Similarly, we can calculate the main effect indices by the K-L method as given in Equation 23.

$$KL_{x_1} : KL_{x_2} : KL_{x_3} = 0.8552 : 0.2692 : 0.0423 = 20.1970 : 6.3582 : 1 \tag{23}$$

The results in Equation 23 can be regarded as the main effect indices by looking in a reverse way, i.e., the smaller the value of $KL_{\sim x_i}$, the large a main effect is. For this example, the relative ranking based on the main effects is the same as that based on the total effects, as observed from Figures 3 and 4. Note that the ranking is also consistent with what we observe from the mathematical structure of the response model. Both Sobol' and the K-L methods give the same importance ranking. It needs to be noted that the ratio between the total effects by the K-L method is different from that by the Sobol' approach. We do not expect the two results to be the same because the Sobol's method, a variance-based method evaluates the impact on the variance while the relative entropy based method measures the divergence between two PDFs.

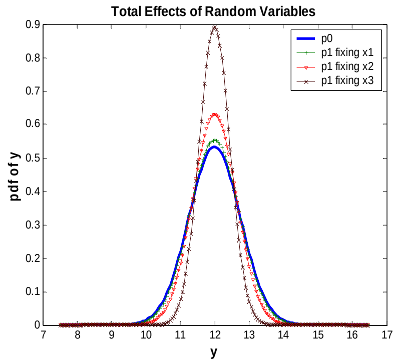

Figure 3. The total effects indices by the K-L entropy based method

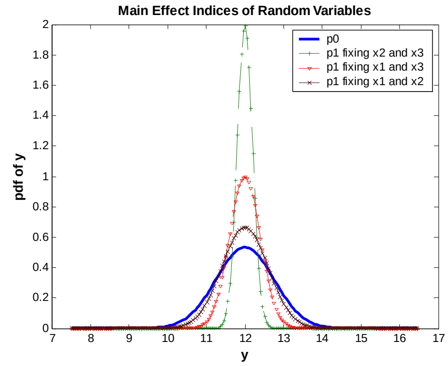

Figure 4. The main effects indices by the K-L entropy based method

2. *GRPSA—Nonlinear Model (highly skewed distribution)*

We consider here another simple nonlinear model $y = x_1/x_2$, where X_1 and X_2 both follow χ^2 distributions with degrees of freedom of 10 and 13.978, respectively, shown in Figure 5. It can be seen from Figure 6 that the distribution of Y is highly-skewed with a long right tail. The impacts of uncertainties in X_1 and X_2 on the distribution of the response are illustrated in Figure 6. The total effect indices of the two variables from our K-L entropy based method and the Sobol' method are compared in Table 2. From Table 2, it is noted that X_1 is more important than X_2 based on the relative entropy. However, the Sobol' method shows that X_1 and X_2 are equally important. The graphical illustration of the divergence of the PDF curves indicates that the effect of X_1 on the whole distribution of Y is higher, which means that the results from the relative entropy method are more trustworthy. This example shows that since the Sobol' method only evaluates the second moment of a distribution. It is no longer a good measure of distinction for highly skewed and heavily-tailed distributions. This example shows the first limitation of the variance-based methods discussed in Section III.A. That is, it demonstrates that the K-L entropy-based method captures more complete information about the uncertainty in a random variable.

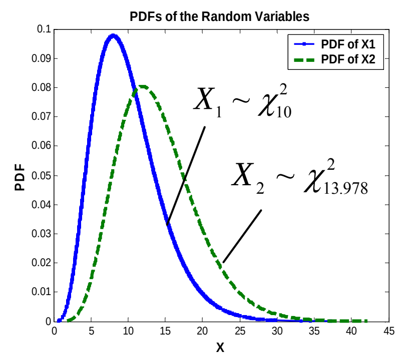

Figure 5. Distributions of random variables

Table 2. Comparison of the total effect indices

		X_1	X_2
KL entropy Based Method	$D_{KLx_i}(p_1 \mid p_0)$, Total Effect	0.1571	0.0791
Sobol's Method	Total Effect Variance, V_{Ti}	0.1676	0.1677
	Total Effect Indices, S_{Ti}	0.5462	0.5465

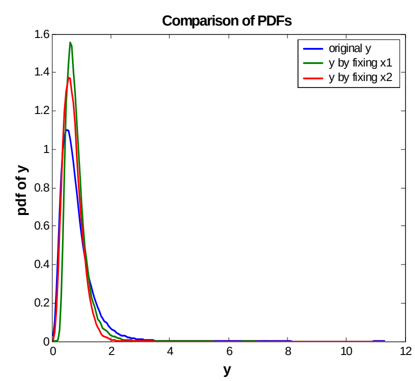

Figure 6. Comparison of total effects by the K-L based Method

B. Engineering Design Problems

1. RRPSA for the Reliability-based Design of a Speed Reducer

A well-known speed reducer problem represents the design of a simple gear box (shown in Figure 7) which is frequently used in many transmission systems such as in a light airplane between the engine and propeller to allow each to rotate at its most efficient speed. It was first modeled by Golinski[25] as a deterministic optimization problem. The objective is to minimize the volume of the device (hence, its weight) while satisfying a number of constraints imposed by gear and shaft design practices. That problem has also been used as a multidisciplinary design example with the coupling between gear design and shaft design disciplines[26]. Du[27] reformulated the problem as a probabilistic design problem by considering uncertainties in some variables, such as the sizes of the components, e.g., the shafts.

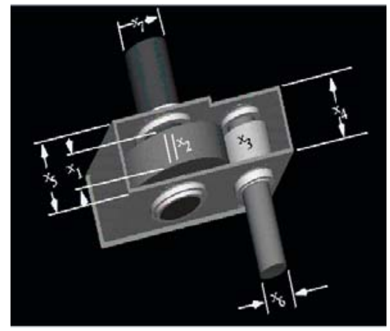

Figure 7. The speed reducer of a small aircraft engine

For a reliability-based design, the objective is to minimize the mean value of the speed reducer volume while satisfying probability requirements of constraints satisfaction. There are two deterministic design variables, five random design variables, and 15 random parameters. It is assumed that all random variables follow normal distribution. There are ten probabilistic constraints plus one deterministic constraint. The formulation of the reliability-based design is provided in Du's dissertation[27]. The required reliability is 95% for all ten probabilistic constraints. At the optimum solution, the optimum volume is 3.3457e+3cm^3. There are five active constraints: g_5, g_6, g_7, g_9, and g_{10}, whose limit state functions are shown in Figure 8. g_5 and g_6 are stresses constraints of the two shafts. g_7, g_9, and g_{10} are the geometric restrictions between components. We use **X** to denote a vector of random design variables (control factors), **X**=[X_1,X_2,…,X_n] and **P** for a vector of random parameters (noise factors), **P**=[P_1,P_2,…,P_k].

$$g_5 = 1.0 - \frac{\sqrt{\dfrac{p_6^2 x_3^2}{d_1^2 d_2^2} + p_7}}{p_5 p_8 x_4^3}$$

$$g_6 = 1.0 - \frac{\sqrt{\dfrac{p_6^2 x_3^2}{d_1^2 d_2^2} + p_9}}{p_{10} p_8 x_5^3}$$

$$g_7 = 1.0 - \frac{p_{11} d_1}{x_1}$$

$$g_9 = 1.0 - \frac{p_{13} x_4 + p_{15}}{x_2}$$

$$g_{10} = 1.0 - \frac{p_{14} x_5 + p_{15}}{x_3}$$

Figure 8. Limit state functions of the active probabilistic constraints

In reliability-based design, for a post-design PSA, the goal is to identify those random variables which have the largest influence on the reliability of performance. The PSA information can be especially beneficial if the required reliability level could not be met via the original reliability-based design. The RRPSA can help designers determine by controlling which random variable, the largest improvement on the reliability could be expected. For this problem, three methods that are applicable for RRPSA are utilized for the five active limit states. The results obtained by Wu's sensitivity method, the MPP-based sensitivity factors, and the K-L entropy based methods for each active probabilistic constraint are compared in Table 3. The most critical variables for each constraint are shown in bold.

Table 3. Comparison of the effects of random variables on the reliabilities of active constraints

Constraint Index	Random Variable	Wu's sensitivity coefficients*		MPP-based method	K-L entropy based method
		S_μ	S_σ	S_i	KL_{x_i}
g_5	X_2	0.0008	0.0145	0.13e-9	0.0000
	X_4	0.0272	0.0038	0.44e-4	6.784e-5
	P_5	**1.3906**	**1.5849**	**0.73428**	**7.929e-1**
	P_6	0.0053	0.0031	0.61e-4	2.984e-5
	P_7	0.6387	0.3162	0.11836	1.1195e-2
	P_8	1.3848	1.5695	0.14725	1.5672e-2
g_6	X_3	0.0012	0.0042	0.46e-11	2.639e-8
	X_5	0.0035	0.0079	0.49e-4	5.098e-5
	P_6	0.0042	0.0069	0.30e-5	1.911e-5
	P_8	**1.3372**	**1.4452**	**0.42416**	**7.460e-2**
	P_9	1.1728	1.0660	0.31446	4.202e-2
	P_{10}	1.0512	0.8965	0.26133	3.038e-2
g_7	X_1	0.0111	0.0060	0.327e-3	0.3772e-4

	P_{11}	**2.0628**	**3.3939**	**0.99997**	∞
g_9	X_2	0.0046	0.0201	0.14e-4	2.651e-5
	X_4	0.0156	0.0010	0.36e-4	5.575e-5
	P_{13}	**2.0577**	**3.3780**	**0.99501**	∞
	P_{15}	0.1458	0.0239	0.494e-2	5.585e-4
g_{10}	X_3	0.0306	0.0177	0.246e-3	8.435e-5
	X_5	0.0422	0.0046	0.307e-3	1.338e-4
	P_{14}	**1.9691**	**3.0937**	**0.91052**	**9.1206**
	P_{15}	0.6129	0.3014	0.08893	8.174e-3

It is observed that the importance rankings of random variables for each constraint are consistent when using three RRPSA methods, even though the mathematical definitions of these methods are different. There are some differences in the relative importance of a few insignificant variables ranked by either S_μ or S_σ when using Wu's sensitivity coefficients. For example, the mean of X_2 is less important than the mean of X_4 for the reliability of g_5. However, the variance of X_2 has larger impact than that of X_4. The reason is that Wu's sensitivity coefficient shows the first-order effect of one distribution parameter only. Both S_θ's should be looked together to tell the effect of a random input. .

It is noted that there are two "infinity" values for total effects when using the K-L based method. One is the total effect of P_{11} on g_7. Another is the total effect of P_{13} on g_9. When uncertainty in a response almost comes from one random resource; therefore, eliminating uncertainty in that random variable nearly removes all uncertainty in the response (illustrated in Figures 9 and 10). In that case, p_1 in Equation 15 has a very large value at the mean of the dominant random variable, while goes to zero elsewhere. This causes the value of KL_{x_1} becomes very large or approaching infinity.

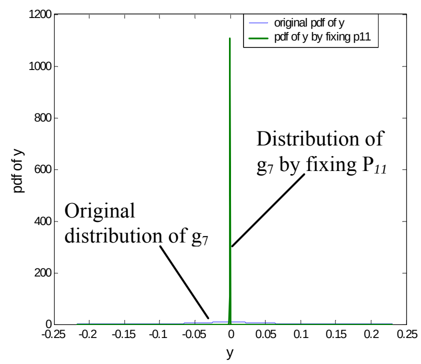

Figure 9. Impact of P_{11} on g_7

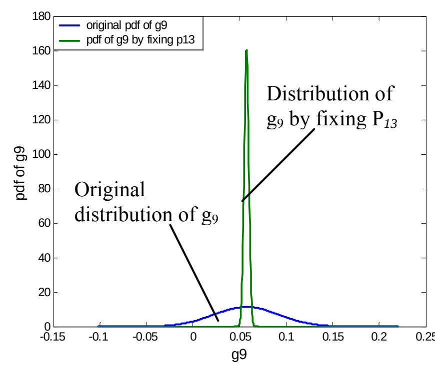

Figure 10. Impact of P_{13} on g_9

The importance rankings obtained are verified by the improvement on reliability through uncertainty reduction in each random variable (shown in Figures 11-15).

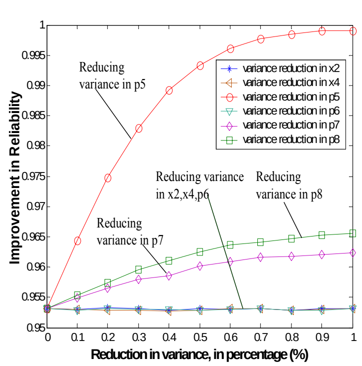

Figure 11. Reliability improvement of g_5 by variance reduction

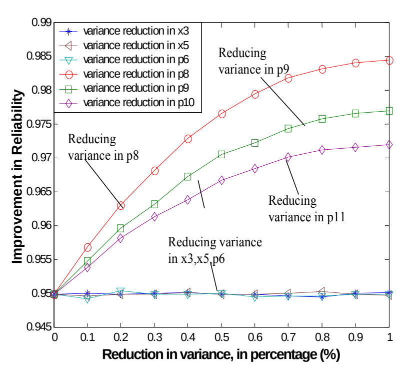

Figure 12. Reliability improvement of g_6 by variance reduction

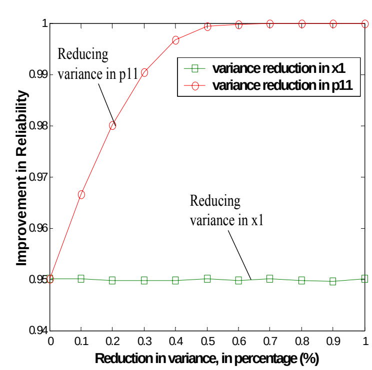

Figure 13. Reliability improvement of g_7 by variance reduction

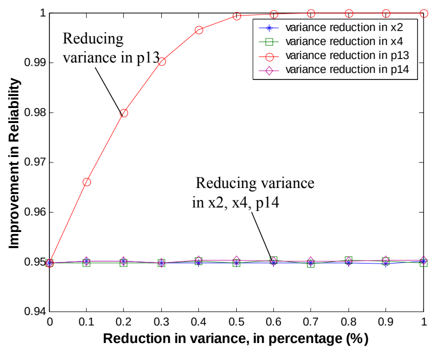

Figure 14. Reliability improvement of g_9 by variance reduction

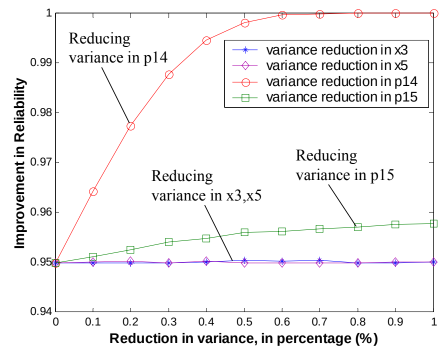

Figure 15. Reliability improvement of g_{10} by variance reduction

From Table 3, it is noted that for g_5, the random parameter P_5 is the most critical. It is confirmed in Figure 11 that by reducing the same amount of variance, P_5 leads to the largest reliability increase than any other variables. For example, if we reduce the uncertainty in P_5 by 10%, the reliability of g_5 can increase from 95% to 96.5%. However, for those insignificant variables, such as X_2, X_4, and P_6, even if all the uncertainties are removed, there is little impact on the reliability of g_5. Figure 14 confirms that P_{13} is the most important uncertainty source for g_9. It can be seen that if reducing 50% of its variance, the reliability of g_9 can reach as high as 1. Even if removing all uncertainties in X_2, X_4, and P_{14}, there is almost no improvement on the reliability of g_9. All the importance rankings listed in Table 3 are consistent with those observed from Figures 11-15. Therefore, all three methods are effective for the post-design RRPSA in this example.

2. Prior-design PSA for the Integrated Reliability and Robust Design of the Speed Reducer

In this example, we show the application of the Sobol' method and the K-L based approach prior to a probabilistic design to reduce the dimension of a design space. The design problem is formulated in Figure 16. In which, **d** is a vector of deterministic design variables, **X** is a vector of random design variables, and **P** is a vector of random parameters. The subscript *L* and *U* denotes the lower and upper bounds for design variables, respectively. w_1 is a weighting factor. $\mu_{volume,\min}$ and $\sigma_{volume,\min}$ are obtained through optimization by setting w_1=1 and 0, respectively.

$$f = w_1 \frac{\mu_{volume}}{\mu_{volume,\min}} + (1-w_1)\frac{\sigma_{volume}}{\sigma_{volume,\min}}$$
$$s.t.,\quad P\{g_j(\mathbf{d},\mathbf{X},\mathbf{P}) \ge 0\} \ge R_j,\quad j = 1,\ldots,m.$$
$$\mathbf{d}_L \le \mathbf{d} \le \mathbf{d}_U$$
$$\mathbf{x}_L \le \mathbf{x} \le \mathbf{x}_U$$
$$where,$$
$$Volume = 0.7854\, x_1 d_1^2 (3.3333\, d_2^3 + 14.9334\, d_2 - 43.0934) - 1.5079\, x_1 (x_4^2 + x_5^2) + 7.477 (x_4^3 + x_5^3) + 0.7854 (x_2 x_4^2 + x_3 x_5^2)$$

Figure 16. Formulation of the integrated reliability and robust design

Our goal is to identify those variables which are not important to both the objective function (volume) and the reliability constraints. For the robust objective, we use the Sobol' and the K-L entropy based methods for GRPSA to identify the insignificant variables. The design objective, the volume of the speed reducer, is a function of design variables only. In the prior-design stage, since the design solution is not yet obtained, all design variables are assumed to follow uniform distributions over their allowable ranges in PSA. Results are listed in Table 4.

Table 4. Comparison of the effects of variables on volume

Objective Function	Random Variable	Total Effects		Main Effects	
		Sobol's Indices S_{Ti}	K-L based KL_{x_i}	Sobol's Indices S_i	K-L entropy $KL_{\sim x_i}$
Volume	d_1	0.9530	0.2156	0.5255	0.4982
	d_2	0.2042	0.0015	0.0109	2.0248
	X_1	0.3336	0.0542	0.0191	0.7101
	X_2	2.99e-6	2.75e-5	0.0015	4.2958
	X_3	5.66e-6	6.4e-4	0.0015	4.3038
	X_4	0.0011	0.1274	0.0026	1.2232
	X_5	0.0042	0.1292	0.0057	1.2233

In the above table, the importance ranking is the same from both methods for both the total effects and the main effects. It should be noted that the smaller the value of $KL_{\sim x_i}$, the more important a variable is. d_1, X_1, and d_2 are the first three most influential design variables. The most insignificant variables are identified as X_2 and X_3.

Using the RRPSA methods, we can generate the importance rankings of random variables for all probabilistic constraints. However, in a prior-design stage, since the mean locations of design variables are not available, we need to perform RRPSA at multiple points in the design space to assess the overall impact of a random variable on a reliability constraint. If a variable is not important at all tested design points, then it is considered as not important for the reliability constraint (g_j). Based on the results of using the K-L entropy-based RRPSA at multiple points in the design space, it is found that X_2 and X_3 are not important for all the probabilistic constraints. Therefore, X_2 and X_3 are treated as deterministic design variables. Using this approach, the size of the problem for the integrated reliability and robust design model is reduced. If we set the weighting factor w_1 as 0.5 and solve for both the original model and the reduced one, we obtain exactly the same optimum solution (f^*=1.0048 with μ_{volume}=3.3457e+3 and σ_{volume}=295.0) and the same optimum point ([$\mathbf{d}^*$, $\mathbf{X}^*$]=[0.7, 17.0, 3.7879, 7.7378, 8.0845, 3.5954, 5.5270]). This verifies that X_2 and X_3 are indeed insignificant random variables which have little impact on the design solution. This example shows that the relative entropy-based approach and the Sobol' method can be used for GRPSA on design objective in the prior-design stage, while the relative entropy-based approach also has the flexibility for studying the variable impact on probabilistic constraints.

C. Conclusion

Probabilistic sensitivity analysis (PSA) is a useful tool in design under uncertainty by providing valuable information of the impact of uncertainty sources on the probabilistic characteristics of a response. In the prior-design stage, PSA can be used to identify those probabilistically insignificant variables and to reduce the design problem dimension. Using this approach, the design efficiency can be improved without much sacrifice on the optimum solution. For post-design analysis, probabilistic sensitivity analysis can be used to identify where to spend resources for potential improvement of a performance by reducing the uncertainty of significant variables. In this work, we review the existing probabilistic sensitivity analysis methods. Based on the range of interest of a performance distribution, they could be classified into two categories: the global response probabilistic sensitivity analysis (GRPSA) and the regional response probabilistic sensitivity analysis (RRPSA). Four alternative PSA methods are selected for detailed comparison. The merits behind each method, their advantages, limitations, and applicability are discussed in details. Variance-based methods, such as Sobol' indices, can only be used for GRPSA. Wu's sensitivity coefficients and the MPP-based sensitivity factors are utilized for RRPSA only. Kullback-Leibler relative entropy based method is applicable for both GRPSA and RRPSA.

Demonstrated by two numerical examples as well as two engineering design problems, we show how to apply the existing PSA methods under various design scenarios at different design stages. For GRPSA, we observe that the K-L entropy based method provides better quantification of the impact of a variable by incorporating of the complete PDF information of a response variable, while the Sobol' method captures only the impact on the response variance (second moment). For RRPSA in the post-design stage of reliability-based design, we show that the K-L entropy-based method provides the effects of all uncertainty in a random variable on the distribution of a random response within the failure region, while Wu's coefficient indicates the first order effect of a distribution parameter on the probability of failure, and the MPP-based approach shows the impact of all uncertainty in a random variable on a specific reliability. In many cases, the rankings obtained from different methods are consistent. When the rankings are different from different methods, it is critical to understand the definitions behind each method to interpret the sensitivity results in the right way. Overall, the K-L entropy based method has the flexibility for both GRPSA and RRPSA in both prior-design and post-design stages. Therefore, the K-L entropy based method can be used as a unified measure in all cases. The results obtained in this study are expected to help designers choose an appropriate PSA method for a specific design problem

Acknowledgments

We are grateful for the support from Ford University Research Program (URP) and the grant from National Science Foundation, DMI 0335877.

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