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RELATIVE ENTROPY BASED METHOD FOR GLOBAL AND REGIONAL SENSITIVITY ANALYSIS IN PROBABILISTIC DESIGN

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ABSTRACT

To overcome the limitations of existing variance-based methods for Probabilistic Sensitivity Analysis (PSA) in design under uncertainty, a new PSA approach based on the concept of relative entropy is proposed. The relative entropy based method evaluates the impact of a random variable by measuring the divergence between two probability density functions of a response. The method can be applied both globally over the whole distribution of a performance response (called global response probabilistic sensitivity analysis-GRPSA) and in any regional range of a response distribution (called regional response probabilistic sensitivity analysis-RRPSA). The former is the most useful for studying variable impact on robust design objective, while the latter provides insight into reliability constraints. The proposed method is applicable to both the prior-design and post-design stages, for variable screening and uncertainty reduction, respectively. The proposed method is verified by numerical examples and industrial design cases.

KEY WORDS

probabilistic sensitivity analysis, robust design, reliability based design, total effect, main effect, relative entropy

1. INTRODUCTION

It has been widely acknowledged that uncertainty is inevitable in a product development process. Robust design (Chen, et al., 1996; Du and Chen, 2000) and reliability-based design (Choi and Youn, et al., 2000, Du and Chen, 2002a) are two widely used probabilistic design methods that have gained wide attentions to ensure the quality of a product under uncertainty. Robust design is used to minimize the effect of variations in controllable and/or uncontrollable factors without eliminating the sources of variations, while the reliability-based design has been widely applied to ensure that a system performance meets the pre-specified target with a required probability level. Though it is important to seek the optimal solution in probabilistic optimization, sensitivity analysis is also playing an important role to help designers gain more knowledge of the complex model behavior and make informed decisions regarding where to spend the engineering effort.

In deterministic design, sensitivity analysis is used to find the *rate of change* in the model output by varying input variables *one at a time* near a given central point, which involves partial derivatives and often called *local sensitivity analysis*. For design under uncertainty, sensitivity analysis has different meanings. The term *probabilistic sensitivity analysis* (PSA) is used because sensitivity analysis is performed with respect to the probabilistic characteristics of model inputs and outputs. In general, *the **probabilistic sensitivity analysis (PSA)** is a study to quantify the impact of uncertainties in random variables on the uncertainty in the model output*.

Among existing probabilistic sensitivity analysis methods, a popular category is the so-called variance-based methods for global sensitivity analysis (Sobol', 1993, 2001; Chan, et al., 1997; Saltelli, et al., 1999, 2000; Jansen, 1999; MacKay et al., 1999). Based on the decomposition of the total variance of an output to various sources, sensitivity indices are defined as the global measures of the output variability over the entire range of the input variables. The variance-based methods can be used to study the impact of different variables on a performance where the input variables are described probabilistically. Obviously, variance-based methods can be applied directly into robust design problems where a part of the design objective is to reduce the performance variance across the whole range of performance distribution. However, those methods can not help

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evaluate the effect of a random variable over a specific partial region of performance distribution, such as the failure region in the reliability-based design. The distinction is important because variables that have the most impact globally may not have the same criticality in the local region of failure, and vice versa. Furthermore, all variance-based methods share the same assumption that the second moment (performance variance) is sufficient to describe the uncertainties encountered. Such assumption may not always be valid, especially in the case where the performance distribution is highly skewed due to highly nonlinear functions or inputs with heavy-tailed distributions.

In this paper, we propose a relative entropy based PSA method that can be adapted to both global and regional sensitivity analyses and overcome the aforementioned limitations of variance-based methods. Entropy (Cover and Thomas, 1991) as a measure of uncertainty associated with a random variable is a popular concept in the information theory literature. Entropy-based methods for sensitivity analysis of decision making have been investigated by several authors (Felli and Hazen, 1998). Ebrahimi, et al. (1999), made a comparison between entropy and variance in ordering univariate distributions. Mutual entropy, as a measure of uncertainty reduction, has been applied in the epistemic sensitivity analysis (Krzykacz-Hausmann, 2001; Frey and Patil, 2002). However, the evaluation of the mutual entropy is not quite straightforward and may experience technical difficulties in its implementation. There are many ways to define entropy measures (see Jumarie, 1990); here, we adopt relative entropy measure which is commonly used as a divergence measure from one probability distribution to another. This concept is used in this work to capture the total impact of the input uncertainty on that of the output both globally and regionally.

The proposed method can be applied to different formulations of probabilistic design. In robust design, it can be used to identify random variables with the most contribution to the variation of a system response. In the reliability-based design, it can be used to identify random variables with the largest impact on the reliability for given probabilistic constraints. Our work also shows that the proposed method can provide valuable information at different design stages. It can be used in the prior-design stage to screen out those variables that are probabilistically insignificant. It can also be applied in the post-design stage to determine the source of uncertainty reduction that will result in the most significant quality improvement.

2. TECHNICAL BACKGROUND

2.1 Variance-Based Methods for PSA

Similar to the concept as used in ANOVA, many of the variance-based methods decompose the total variance of an output $Y=h(\mathbf{X})$ to items contributed by various sources of input variations $\mathbf{X}=[X_1,X_2,\ldots,X_n]$, and then derive sensitivity indices as the ratios of a partial variance contributed by an stochastic input variable to the total variance of the output. Details of ANOVA decomposition for sensitivity analysis are provided in Chen et al. 2004. In brief, the variance (V) of a response is decomposed into a summation of functions through an ANOVA-like way:

$$V = \sum_i V_i + \sum_{i<j} V_{ij} + \ldots + V_{1,2,3,\ldots,n} \quad (1)$$

In Equation 1, V is the total variance in the model output. The first order terms V_i represent the partial variance in Y due to the individual effect of a random variable X_i, while the higher order terms show the interaction effects between two or more random variables. For example, V_{ij} is the partial variance in Y due to the interaction between two random variables X_i and X_j.

The main effect index of a random variable X_i is obtained by the normalization of the main-effect variance over the total variance in Y as shown in Equation 2. Equation 3 gives the sensitivity index for the interaction between two random variables, X_i and X_j. A general sensitivity index is given in Equation 4.

$$S_i = V_i / V, \quad 1 \le i \le m. \quad (2)$$

$$S_{ij} = V_{ij} / V, \quad 1 \le i < j \le m. \quad (3)$$

$$S_{i,\,i+1,\ldots,m} = V_{i,\,i+1,\ldots,m} / V \quad (4)$$

When there is a large interaction between random variables, evaluation of main effects only is not enough. In such situation, the *total effect* of a random variable X_i, which includes its main effect and all the interaction effects involving X_i, is required to accurately describe its contribution. The total effect index is a useful summary measure when there are a large number of interaction terms for which their computations and interpretations may not be practical. By partitioning the variables to a subset of interest and its complementary, i.e., X_i and $\mathbf{X}_{\sim i}$, where the latter is the subset of all variable excluding X_i, then the total effect index (TSI) is given by

$$S_{Ti} = 1 - S_{\sim i} \quad (5)$$

Many methods have been proposed to calculate the above main and total effect indices (including their variations). Most of the computation methods are sampling-based methods such as correlation ratio, various importance measures, and Sobol'. Jensen (1996) proposed alternative Monte Carlo approaches for the main and total effect indices. Another method, FAST, provides a new way to evaluate a variance by converting multi-dimensional integral to a one-dimensional integral. Saltelli, et al. (1999) showed that sensitivity indices from FAST are equivalent to those from Sobol', but with better computation efficiency. Chan, et al. (2000), showed that Jensen's approach is better for the total effect while Sobol' method is better for the main effect. In Chen et al. 2004, analytical formulations are derived via the commonly used metamodels such as polynomial function and Kriging models. Since variance-based methods only study the impact of a probabilistic input on the variability (first and second moments) of an output, their use in design under uncertainty can be limited. In this work, we propose a PSA that suits various probabilistic design scenarios.

2.2 Probabilistic Design Scenarios

In the context of design under uncertainty, *Probabilistic Sensitivity Analysis (PSA)* is the study of the impact of uncertainties in both controllable and uncontrollable variables (i.e., control and noise factors, respectively). The word *impact* has different meanings under different probabilistic design scenarios. For robust design, PSA is to identify those random variables which contribute the most to the entire distribution (e.g., variance) of a response *y*. For reliability-based design, it means the contributions of different sources of uncertainty to the probability of probabilistic constraints satisfaction, or often called as reliability requirements.

Robust Design	Reliability-Based Design
Find $\vec{\mu}_x$ *Min.* $f = f(\mu_y, \sigma_y)$ *s.t.* $g_j = \mu_{g_j} + k_j\sigma_{g_j} \geq g_{j,L}$ $\vec{d}_L \leq \vec{d} \leq \vec{d}_U$ $\vec{x}_L \leq \vec{x} \leq \vec{x}_U$ $j = 1,2,\ldots,m.$	*Find* $\vec{d}$, $\vec{\mu}_x$ *Min.* $y = h(\vec{d}, \vec{x}, \vec{p})$ *s.t.* $g_j = P\{g(\vec{d}, \vec{x}, \vec{p}) \geq g_{j,L}\} \geq R_j$ $\vec{d}_L \leq \vec{d} \leq \vec{d}_U$ $\vec{x}_L \leq \vec{x} \leq \vec{x}_U$ $j = 1,2,\ldots,m.$

Figure 1. Probabilistic Design Formulations

In the robust design formulation in Figure 1, $\vec{d}$, $\vec{x}$ and $\vec{p}$ are vectors of deterministic design variables, random controllable design variables, and random uncontrollable variables (i.e., noise factors), respectively. μ_y, μ_{g_j}, and σ_y, σ_{g_j} are the mean and standard deviation of the objective and constraint performance, respectively. In the reliability-based design formulation, R_j is the desired probability of constraint satisfaction corresponding to g_j.

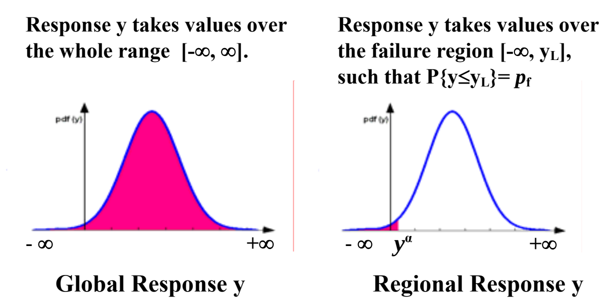

Figure 2. Global vs. Regional Response Range

As shown in Figure 2, in robust design, because the design objective is to reduce the response variation, the full range distribution of a response, i.e., [-∞,+∞] or [y^{α}, y^{β}] where y^{α} and y^{β} are the response at α and β quantiles, respectively—is of interest for PSA. On the other hand, in the reliability-based design, the interest is on assessing the impact on preventing the failure in a partial region, say [-∞,y^{α}]. Accordingly, we define two types of PSA methods:

- **Global response probabilistic sensitivity analysis (GRPSA)** — PSA for the case that the interest is among the entire distribution of a response;
- **Regional response probabilistic sensitivity analysis (RRPSA)** — PSA for the case that the interest is among a partial range of a response distribution.

In the following discussion, we introduce the use of relative entropy as a unified measure to deal with both GRSPA and RRSPA.

2.3 Kullback-Leibler Entropy

The entropy-based measures for sensitivity analysis can be classified into two types. One is to measure the uncertainty reduction in a random output if a random input is perfectly known, such as the mutual entropy (Krzykacz-Hausmann, 2001). The calculation of mutual entropy involves the joint probability density function (PDF) of random variables and thus it can be expensive to compute. The other type of entropy measure is to measure the divergence between the change of a distribution before and after the elimination of a random variable such as Kullback-Leibler (K-L) entropy (Kullback and Leibler, 1951) defined as

$$D_{KL}(p_1 \mid p_0) = \int_{-\infty}^{\infty} p_1(y) \cdot \log \frac{p_1(y)}{p_0(y)} dy = E_{p_1}\left[\log \frac{p_1(y)}{p_0(y)}\right] \quad (6)$$

Kullback-Leibler entropy has the following properties (Csiszar, 1975; Jumarie, 1990; Lin, 1991):

- It is nonnegative. It is zero if and only if p_1 and p_0 are exactly the same.
- It is not a true metric because the measure is not symmetric and it does not satisfy triangle inequality.
- $p \log \frac{p}{0} = \infty$ and $0 \log \frac{0}{p} = 0$

Kullback-Leibler entropy is traditionally used to measure the divergence between two PDFs p_1 and p_0, which are the true and estimated distributions, respectively. The K-L entropy can be interpreted as the expectation of log likelihood of random variable *y* with a PDF of $p_1(y)$. It is noticed that Shannon entropy or differential entropy could be viewed as a special case of Equation 6, when p_0 is a uniform probability density (mass) function. In other words, Shannon entropy, or called differential entropy in continuous case, could be viewed as the divergence from a PDF to a uniform distribution.

3. ENTROPY-BASED PSA METHOD

Suppose a random response Y = *h*(**X**) has a PDF of p_0, where **X** denotes a vector of random inputs, i.e., **X** = [X_1,X_2,…,X_n]. If a random input X_i is fixed to its mean value, or it is perfectly known (i.e., X_i is now treated as a deterministic variable), the PDF of Y becomes p_1. Since X_i becomes a constant now, all impacts of the uncertainty in X_i are eliminated, including both the main effect of X_i and its interaction with other random inputs. Therefore, the relative entropy provides the total effect of X_i on the distribution of Y. In this paper, only the equations for continuous situations are derived. For discrete cases, the integral will be simply replaced by a summation.

For GRPSA, we propose a K-L entropy based method as

$$D_{KLx_i}(p_1 \mid p_0) = \int_{-\infty}^{\infty} p_1(y(x_1,\ldots\bar{x}_i,\ldots x_n)) \cdot \log \frac{p_1(y(x_1,\ldots\bar{x}_i,\ldots x_n))}{p_0(y(x_1,\ldots x_i,\ldots x_n))} dy \quad (7)$$

where, $\bar{x}_i$ means fixing X_i at a value, usually chosen at its mean, μ_{x_i}. Equation 7 measures that over the entire range of *y*, how much the PDF of Y changes after reducing the total variability in X_i. $D_{KLx_i}(p_1 \mid p_0)$ is the total effect index of X_i. The larger the $D_{KLx_i}(p_1 \mid p_0)$, the more important the X_i is.

A measure of the main effect of X_i can be obtained by fixing all other random variables except X_i in p_1, as shown in Equation 8. Because $D_{KL\sim x_i}(p_1 \mid p_0)$ is actually the combined effect of the complimentary of X_i, the $D_{KL\sim x_i}(p_1 \mid p_0)$ can be viewed as the reverse of the main effect of X_i, i.e., the smaller the $D_{KL\sim x_i}(p_1 \mid p_0)$, the more important the main effect of X_i is. It should be noted that $D_{KL\sim x_i}(p_1 \mid p_0)$ itself is not the main effect, but can be used to interpret the main effect.

$$D_{KL\sim x_i}(p_1 \mid p_0) = \int_{-\infty}^{\infty} p_1(y(\bar{x}_1,\ldots x_i,\ldots\bar{x}_n)) \cdot \log \frac{p_1(y(\bar{x}_1,\ldots x_i,\ldots\bar{x}_n))}{p_0(y(x_1,\ldots x_i,\ldots x_n))} dy \quad (8)$$

One advantage of the K-L entropy approach is that the proposed method can also be used for RRPSA, with simple adjustments in the formulae. Because of the reason that will apparent later, we propose a modified K-L entropy so that the measure can be used for both GRPSA and RRPSA. For RRPSA, the integration range of *y* is changed to the region of interest [y_L, y_U], instead of [-∞, +∞] in Equations 7 and 8. The total effect of X_i is defined as

$$D_{KLx_i}(p_1 \mid p_0) = \int_{y_L}^{y_U} p_0(y(x_1,\ldots\bar{x}_i,\ldots x_n)) \cdot \left| \log \frac{p_1(y(x_1,\ldots\bar{x}_i,\ldots x_n))}{p_0(y(x_1,\ldots x_i,\ldots x_n))} \right| dy \quad (9)$$

where, the meanings of p_0, p_1, and x_i are the same as above. y_L and y_U defines the lower and upper bounds of a region of *y*. Different from Equation 7, Equation 9 measures the absolute divergence between two distributions over a region of interest. The absolute value is used here because in a partial region of Y, the log-likelihood can take both positive and negative values at different values of *y*. The absolute value of log likelihood instead of direct use of Equation 7 is introduced because during integral calculation, the positive and negative log-likelihood values may cancel with each other. In that case, the relative entropy could be very small although there is actually large difference between p_0 and p_1. In addition, in Equation 9, p_0 is used as a weighting factor applied in front of the log-likelihood, instead of p_1. The reason is that in a partial region such as the extreme tail of a distribution, the value of p_1 could be very small, even approaching zero, which will diminish the actual effect of divergence. Similar to the GRPSA, Equation 9 measures the total effect of X_i on Y over a range [y_L, y_U]. The larger the $D_{KLx_i}(p_1 \mid p_0)$, the more important the total effect of X_i is.

Similarly, by fixing all the random variables except X_i in p_1 in Equation 9, we get an equivalent but opposite measure of the main effect of X_i over [y_L, y_U], as shown in Equation 10. The smaller the $D_{KL\sim x_i}(p_1 \mid p_0)$, the more important the main effect of X_i is.

$$D_{KL\sim x_i}(p_1 \mid p_0) = \int_{y_L}^{y_U} p_0(y(\bar{x}_1,\ldots x_i,\ldots\bar{x}_n)) \cdot \left| \log \frac{p_1(y(\bar{x}_1,\ldots x_i,\ldots\bar{x}_n))}{p_0(y(x_1,\ldots x_i,\ldots x_n))} \right| dy. \quad (10)$$

In the context of engineering design, PSA can be performed both in the prior-design and in the post-design stage to gain valuable information about the model and its probabilistic behavior.

Prior-Design

The goal of prior-design PSA is aimed to answer the following question:

Which variable(s) could be safely eliminated without bringing much influence on the uncertainty in the response?

The use of PSA in this stage is intended to reduce the size of a problem through variable screening, by generating an importance ranking of all variables. Based on the ranking, unimportant deterministic design variables are fixed at chosen values, i.e., treated as constants. Similarly, probabilistically insignificant random variables could be treated as deterministic variables, i.e., fixed at their mean values. It is noted that in the prior-design stage, we have no idea about the location of the design point. When applying the PSA at this stage, both the deterministic and random (controllable) design variables are considered to be uniformly distributed over their entire range, while random noise variables follow the pre-specified distributions.

Post-Design

Once a design solution is identified through the means of optimization, the focus of the PSA in the post-design stage is to answer the following question:

Which random (noise) uncertainties should be further controlled (eliminated) to gain the largest improvement on the probabilistic performance of a response?

An example of the above case is “tolerance design” where manufacturing precision is improved to reduce or eliminate variations. The post-design PSA is applied to decide and prioritize available resource to reduce the sources of variation. That is, variation control exercise should be spent on the variables with the highest importance because they contribute the most to the output variations. Conversely, tolerance requirements can be relaxed for noise factors with negligible effects to output variation.

For the reliability-based design, a particular interest is on the sensitivity of the probability of failure (p_f) or its complementary – the probability of success (i.e., reliability) (1-p_f). For a given design setting, the system has two possible outcomes: safe or unsafe, with probability of 1-p_f and p_f respectively (Reid, 2002). Applying relative entropy on this discrete event requires the calculation of expected value of entropies of both events. Therefore, the PSA for reliability is calculated based on the contributions of random variables to both the success and failure conditions, denoted as $D_{KLx_i}^{p_f}\left(\hat{p}_f \mid p_f\right)$, as follows:

$$D_{KLx_i}^{p_f}\left(\hat{p}_f \mid p_f\right) = \hat{p}_f \log\frac{\hat{p}_f}{p_f} + \left(1-\hat{p}_f\right)\log\frac{1-\hat{p}_f}{1-p_f} \qquad (11)$$

where, p_f is the original probability of failure of a performance. $\hat{p}_f$ is the probability of failure when fixing X_i at its mean value. Equation 11 is the total sensitivity index of X_i on the reliability. Similarly, by fixing all the random variables except the one of interest in $\hat{p}_f$, $D_{KL\sim x_i}^{p_f}\left(\hat{p}_f \mid p_f\right)$ provides an equivalent measure of the main effect of x_i. We view the formulation in Equation 11 as an alternative approach to the evaluation of uncertainty impact on the reliability formulated in Equation 9.

4. COMPUTATIONAL ISSUES

The evaluations of the proposed K-L entropy measures (Equations 7-11) involve the estimation of two PDFs (p_o and p_1) of a given response: one before and one after variance reduction in inputs. If the response performance is easy to compute, Monte Carlo simulations can be employed. However, in real engineering applications with high-fidelity and expensive simulation models, the PDF estimation via Monte Carlo simulation is impractical or even impossible. Alternative approximation approaches can be used to overcome the computational barrier. One approach is to use the metamodeling techniques (Chen, et al., 2004) to build surrogate models as approximations of high-fidelity models. Sampling techniques are then applied to the easy-to-compute metamodels. Based on samples, the PDF of a random response could be obtained by kernel density estimation (KDE) at any value of *y* defined as

$$\hat{f}(y) = \frac{1}{n\delta}\sum_{i=1}^{n} K\left(\frac{y_i - y}{\delta}\right) \qquad (12)$$

where *K* is a symmetric probability density function; δ is the window width or bandwidth; and *n* is the total number of samples (Sheather and Jones, 1991).

As an alternative to the sampling-based methods, the PDF information could be obtained by the most probable point-based uncertainty analysis (MPPUA) method (Du and Chen, 2001) or the First Order Saddlepoint Approximation (Du and Sudjianto, 2004). The concept of the most probable point (*MPP*) is utilized to generate the cumulative distribution function (*CDF*) of a system output by evaluating probability estimates at a serial of limit states across a range of output performance. The PDF can then be derived as the derivative of the CDF.

In practice, the choice of the metamodeling or the MPPPUA method will depend on the samples required to fit an acceptable metamodel and the computational needs for the MPPs search. In this paper, we use the KDE for PDF estimation based on the samples obtained through simulations, where the normal density is used as the kernel function in Equation 12. For the engineering design problems in Section 5.2, metamodels are used for sampling instead of directly using the computationally expensive simulations. All other examples are mathematical examples with simple analytical equations.

5. VERIFICATIONS AND APPLICATIONS

There are four objectives in our verification study. First, we want to verify whether our proposed entropy based approach for GRPSA will provide consistent results compared with the Sobol' method. We need to bear in mind that we do not expect the ranking of variables from the two methods are exactly the same because our proposed entropy based method shows the impact on the whole PDF of the response which includes higher order moment statistics , while the Sobol' method provides the impact on the response variance (i.e., the second order moment). The second objective is to gain some insight to understand in what situations the two methods will provide distinctively different rankings of variables. The third objective is to verify whether the alternative methods we propose for RRPSA (Equations 9 and 11) indeed provide the right ranking of variables in terms of their impact on improving the performance reliability. We verify our results by comparing the actual reliability improvement by variance-based reduction in each random variable. Finally, we want to examine whether our proposed methods can truly benefit the process of design under uncertainty.

5.1 Numerical Examples

(1) GRPSA for A Nonlinear Model

Lets consider a simple quadratic model $y = x_1 x_2 + x_3^2$ with three independent random variables X_1, X_2, and X_3, all following uniform distributions over [-1,1]. Both the proposed GRPSA method in Equation 7 and the Sobol' method (Sobol, 2001) are applied to determine the main and total effects of each variable. The effects of different random variables are graphically illustrated in Figure 3. The differences between the original PDF curve of Y when all random variables vary and the PDFs of Y by fixing either X_1, X_2, or X_3 indicates the total effects of X_1, X_2, and X_3 respectively. The larger the difference, the higher the impact of a random variable is. The global sensitivity information obtained by both our method and the Sobol' method is listed in Table 1.

We know from the explicit mathematical structure that X_1 and X_2 are symmetric and independent with identical distributions; therefore, they should have the same effect on the response Y. This is evident in Figure 3, where the PDF of Y by fixing X_1 at its mean value coincides exactly with the PDF by fixing X_2 at the mean. Our results of GSA Table 1 match with this feature - both the main and the total effects of X_1 and X_2 are almost the same. For this particular example, both methods provide the same importance ranking of three variables based on either the main or the total effects. However, the ranking based on the main effect is different from that based on the total effect. In particular, the main effect of X_3 is larger than that of X_1 or X_2, however, its total effect is slightly smaller than that of X_1 or X_2. These observations are consistent with the structure of the equation $y = x_1 x_2 + x_3^2$, where the interaction only occurs between X_1 and X_2. We should point out that $D_{KL\sim x_i}$ itself is not the actual main effect. It is an equivalent but opposite measure that should be viewed in the same way as $S_{\sim i}$ in Equation 5. Finally, it should be noted that the K-L entropy based method can only provide the relative but not the absolute importance of random variables. It could not answer absolutely how much uncertainty in Y comes from a random variable X_i. On the other hand, the effect indices from Sobol' method are absolute, normalized measures, or called quantitative measures by Saltelli (2000).

Table 1. Comparison of GRPSA results

		x_1	x_2	x_3
KL Entropy Based PSA	$D_{KL\sim x_i}(p_1 \mid p_0)$ - Main	4.0783	3.9953	0.2917
	$D_{KLx_i}(p_1 \mid p_0)$ - Total	0.3783	0.3783	0.3523
Sobol' D=0.2	Main Effect Variance, V_i	0.0004	0.0003	0.0891
	Main Effect Indices, S_i	0.0019	0.0017	0.4455
	Total Effect Variance, V_{T_i}	0.1112	0.1113	0.0891
	Total Effect Indices, S_{T_i}	0.5560	0.5565	0.4455

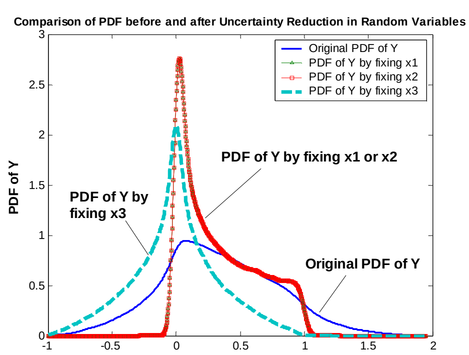

Figure 3. Comparison of impacts of uncertainty

(2) GRPSA on Highly-Skewed Distribution of Y

Lets consider here another simple nonlinear model $y = x_1 / x_2$, where X_1 and X_2 both follow χ^2 distributions with degrees of freedom as 10 and 13.978, respectively, shown in Figure 4. It can be seen from Figure 5 that the distribution of Y is highly-skewed with a long right tail. The impacts of uncertainties in X_1 and X_2 on the distribution of the response are illustrated in Figure 5. The total effect indices of the two variables from our K-L entropy based method and the Sobol' method are compared in Table 2. From Table 2, it is noted X_1 is more important than X_2 based on the relative entropy. However, the Sobol' method shows that X_1 and X_2 are equally important. The graphical illustration of the divergence of the PDF curves indicates that the effect of X_1 is higher, which means that the results from the relative entropy method are more trustworthy. This example shows that since the Sobol' method only evaluates the second moment of a distribution, it is no longer a good measure of dispersion for highly skewed and heavily-tailed distributions.

Table 2. Comparison of the total effect indices

		X_1	X_2
KL entropy Based Method	$D_{KLx_i}(p_1 \mid p_0)$, Total Effect	0.1571	0.0791
Sobol's Method	Total Effect Variance, V_{Ti}	0.1676	0.1677
	Total Effect Indices, S_{Ti}	0.5462	0.5465

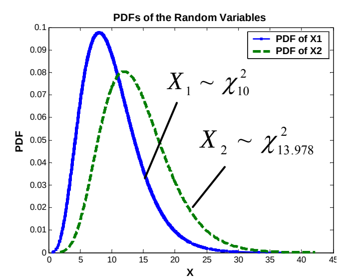

Figure 4. Distributions of random variables

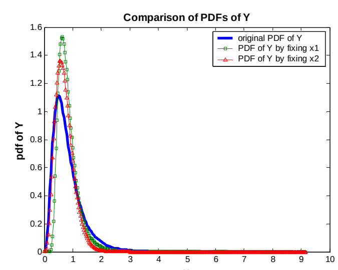

Figure 5. Comparison of impacts

(3) Verification of RRPSA Methods

For the same model in the first example, we apply the RRPSA methods in Equations 9 and 11 for the post-design analysis in the reliability-based design. The three random variables follow a normal distribution with mean of 1.0 and a relatively large variance of 1.0. For a desired reliability 90%, a critical failure mode is defined as $y = x_1 x_2 + x_3^2 \geq 0.068$. We apply both Equations 9 and 11 in the failure region [$-\infty$, 0.068] to identify the probabilistically important variables. We expect that the reduction of uncertainty in the most critical variables would lead to the largest improvement on the reliability.

As observed from Figure 6, in the failure region, fixing X_1 or X_2 will cause larger divergence in the distribution curve of Y than that by fixing X_3. Therefore, X_1 and X_2 are more important than X_3. This observation is confirmed by the sensitivity information listed in Table 3. The results are further verified by calculating the actual reliability improvement. The last row in

Table 3 shows the improvement on the reliability if the uncertainty in a random variable could be eliminated completely. Although usually it is not possible to eliminate completely uncertainty in a random variable, the results are good indications of where to put efforts effectively to improve the reliability.

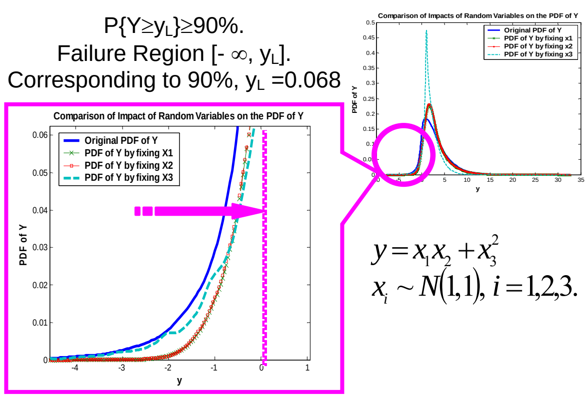

Figure 6. PDF of Y and the enlargement of the left tail

Table 3. Effects of random variables in the failure region

	X_1	X_2	X_3
Eqn. 9: $\int_{-\infty}^{0.068} p_0 \left\lvert \log \frac{p_1}{p_0} \right\rvert dy$	0.1320	0.1324	0.0478
Eqn. 11: $D_{KLx_i}^{p_f}\left(\hat{p}_f \mid p_f\right)$	0.0128	0.0128	0.0074
Reliability improvement after uncertainty reduction	4.43%	4.43%	3.44%

5.2 Engineering Design Problems

(1) Robust Design for Engine Block and Head Joint Sealing Assembly

Engine block and head joint sealing assembly is one of the most crucial structural designs in the automotive internal combustion engine. As shown in Table 4, there are six deterministic design variables (x_1-x_6), and two random noise factors (P_1 and P_2) following normal distributions. For the confidentiality reason, the values of all variables and noise factors are normalized within [1,3]. The robust design objective is to minimize the *gap lift* of the assembly (x_1-x_6) as well as its sensitivity to manufacturing variation (P_1 and P_2). The design is very complex involving multiple components with complicated geometry. To reduce the computational cost, a Kriging model is created for the gap lift based on the data from computer experiments.

Table 4. Design variables and parameters

	Meaning	Lower limit	Upper limit	μ	σ
x_1	Gasket thickness	1	3	–	–
x_2	Number of contour zones	1	3	–	–
x_3	Zone-to-zone transition	1	3	–	–
x_4	Bead profile	1	3	–	–
x_5	Coining depth	1	3	–	–
x_6	Deck face surface flatness	1	3	–	–
P_1	Load/deflection variation	1	3	2	0.33
P_2	Head bolt force variation	1	3	2	0.33

We use this example to illustrate the effectiveness of the proposed K-L entropy based method (Equation 7) to reduce the dimensionality of a design problem in the prior-design stage. Since the optimal solution in unknown, we consider design variables (x_1-x_6) following uniform distribution within their allowable range. The noise variables, P_1 and P_2 , follow their pre-specified (normal) distributions. The sensitivity information obtained by our proposed K-L entropy-based method is shown in Table 5, and compared to that from the Sobol' method shown in Table 6. The importance ranking of all variables is shown in Figure 7.

Table 5. Main and total effects from by KL based method

	x_1	x_2	x_3	x_4	x_5	x_6	P_1	P_2
Main Effect $D_{KL\sim x_i}$	0.4589	0.406	6.523	11.030	3.300	0.647	9.589	2.559
Total Effect D_{KLx_i}	0.3334	0.121	0.0	0.0	0.0	0.195	0.0	0.006

Table 6. Main and total effects from Sobol' method

	x_1	x_2	x_3	x_4	x_5	x_6	P_1	P_2
Main Effect V_i	0.203	0.020	0.001	0.001	0.004	0.081	0.001	7.51e-5
Total Effect V_{Ti}	0.351	0.064	1.66e-6	2.03e-6	0.010	0.248	6.47e-7	0.007

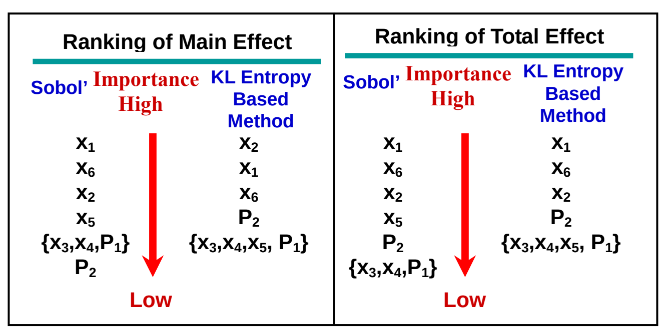

Figure 7. Importance ranking of engine design

It is observed that the rankings of the total effects are generally consistent by two methods, except the relative importance between x_5 and P_2. Note that x_3, x_4, and P_1 are not important from both the main and the total effects. Therefore, we can fix x_3 and x_4 as constants and P_1 at its mean. The model for robust design is reduced to searching the values of four deterministic design variables (x_1, x_2, x_5, and x_6) subject to only one random noise parameter, P_2. The robust design formulation is shown in Figure 8.

$$
\begin{aligned}
&Find \quad \mu_{x_i} \\
&Min. \quad f = w\frac{\mu_y}{\mu_{y\_\min}} + (1-w)\frac{\sigma_y}{\sigma_{y\_\min}} \\
&s.t. \quad \vec{x}_L \le \vec{x} \le \vec{x}_U \\
&\qquad j = 1,2,\ldots, m.
\end{aligned}
$$

Figure 8. Robust design formulation for engine design

In the above formulation, $\mu_{y\_\min}$ and $\sigma_{y\_\min}$ are obtained by setting the weighting factor at $w=1$ and 0, respectively. For both the original model and the reduced model, w is set to be 0.5. Various starting points are tried for better chance to reach the global optimum. When using the original robust design model, the optimum solution is f^*= 1.0872 with $\mu_y = 2.0580$ and $\sigma_y = 1.8e-3$. The optimum point is **x***=[1.7774 1.0 1.9557 3.0 1.0317 2.9387]. For the reduced model by fixing x_3, x_4, and P_1, the optimum solution is f^*= 1.0818 with $\mu_y = 2.1003$ and $\sigma_y = 5.0719e-5$. The optimum point is **x***=[1.7522 1.0726 2.0043 2.0 2.0 2.9104]. It is noted that using the reduced model, the robust design reaches almost the same solution as using the original model, but with much smaller variance of the objective. The optimum points are different. This example shows that the complexity of a robust design problem can be reduced with little sacrifice by using the GRPSA method proposed in this work for variable screening.

(2) Reliability-Based Design for the Vehicle Crashworthiness of Side Impact

We apply the proposed RRPSA methods, Equation 9 and 11, for probabilistic sensitivity analysis in the post-design stage of a reliability-based design for vehicle crashworthiness of side impact (Du and Chen, 2004). The design problem has 11 random variables, among which 9 are design variables and the other two are noise factors. The objective is to minimize the weight of the structure. Detailed descriptions of the reliability-based design problem could be found in Du's dissertation (2002). The required reliability is 99.865% for all ten probabilistic constraints. At the optimum solution, f^*=28.4397 kg, and there are three active probabilistic constraints: g_2, g_8, and g_{10}, whose limit state functions are shown in Figure 9. For the post-design PSA, the goal is to identify variables whose variation has the most impact on the reliability. If possible, by controlling the uncertainty in those critical variables, larger improvement on reliability is expected. The total effects based on $D_{KL_{X_i}}(p_1 \mid p_0)$ and $D^{p_f}_{KL_{X_i}}(\hat{p}_f \mid p_f)$ are shown in Table 7.

$$
\begin{aligned}
g_2 &= 1.0 - \frac{46.36 - 9.9x_2 - 12.9x_1x_8 + 0.1107x_3p_1}{32} \\
g_8 &= 1.0 - \frac{4.72 - 0.5x_4 - 0.19x_2x_3 - 0.0122x_4p_1 + 0.009325x_6p_1 + 0.000191p_2^2}{4.01} \\
g_{10} &= 1.0 - \frac{16.45 - 0.489x_3x_7 - 0.843x_5x_6 + 0.0432x_9p_1 - 0.0556x_9p_2 - 0.000786p_2^2}{15.69}
\end{aligned}
$$

Figure 9. Limit state functions of three active constraints

The importance ranking based on total effects in Table 7 is shown in Figure 10. The rankings of random variables for three active constraints are verified by the improvement of reliability of the three through uncertainty reduction in random variables (shown in Figure 11-13).

Table 7. Total effects of random variables on three active constraints using RRPSA methods

	g_2		g_8		g_{10}	
	$D_{KL_{X_i}}(p_1 \mid p_0)$	$D^{p_f}_{KL_{X_i}}(\hat{p}_f \mid p_f)$	$D_{KL_{X_i}}(p_1 \mid p_0)$	$D^{p_f}_{KL_{X_i}}(\hat{p}_f \mid p_f)$	$D_{KL_{X_i}}(p_1 \mid p_0)$	$D^{p_f}_{KL_{X_i}}(\hat{p}_f \mid p_f)$
X_1	0.0003	4.215e-6	—	—	—	—
X_2	0.0012	2.451e-4	0.0001	1.482e-7	—	—
X_3	0.0002	2.134e-6	0.0001	3.713e-8	0.0001	3.36e-7
X_4	—	—	0.0001	4.600e-7	—	—
X_5	—	—	—	—	0.0021	4.887e-4
X_6	—	—	0.0001	2.748e-6	0.0002	2.904e-5
X_7	—	—	—	—	0.0001	1.849e-6
X_8	0.001	6.016e-7	—	—	—	—
X_9	—	—	—	—	0.0001	1.492e-7
P_1	0.4522	1.346e-3	0.4522	9.004e-4	0.0550	1.351e-3
P_2	—	—	0.0156	9.990e-4	0.0002	5.506e-6

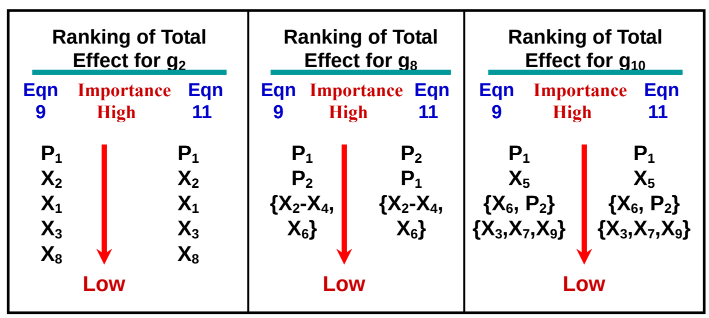

Figure 10. Importance ranking of total effects

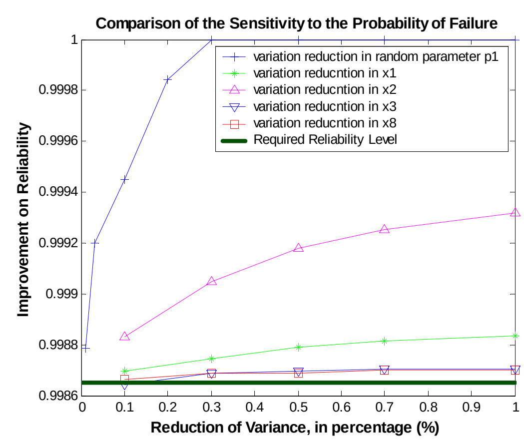

Figure 11. Reliability improvement by variation reduction in random variables – g_2

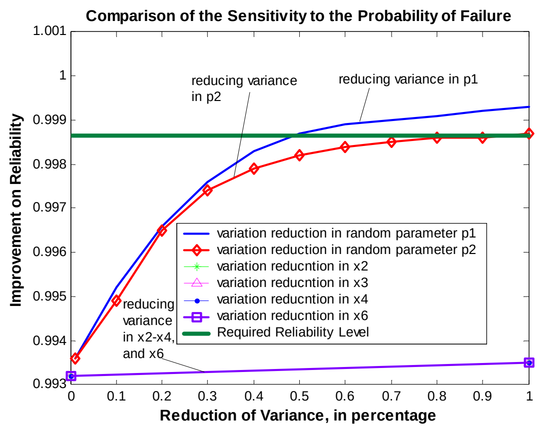

Figure 12. Reliability improvement by variation reduction in random variables – g_8

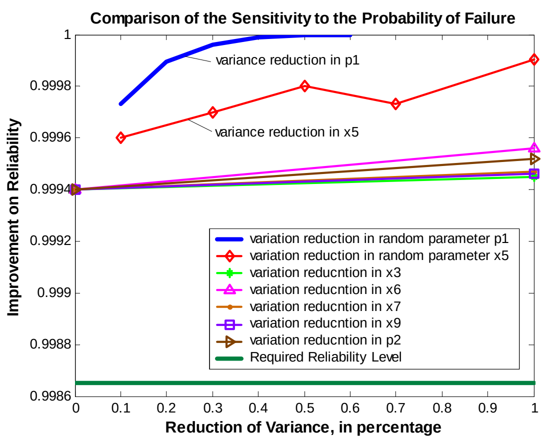

Figure 13. Reliability improvement by variation Reduction in random variables – g_{10}

It is noted from Table 7 that P_1 is ranked as either the first or the second most important variable for all three probabilistic constraints. Its total effects on the reliability for each limit state are confirmed in Figures 11 to 13, where different percentages of variation reductions are tested. By reducing the same amount of uncertainty, P_1 will lead to the largest reliability increase than any other random variables. The inconsistent ranking from Equations 9 and 11 for g_8 (as shown in Figure 10) is caused by the numerical error of using Equation 11 for assessing RRPSA at the extreme tail. It is noticed in Table 7 that the values of $D_{KLx_i}^{p_f}\left(\hat{p}_f \mid p_f\right)$ in Equation 11 are quite small for insignificant variables such as X_1, X_3, and X_5 in g_2. Considering the numerical errors introduced in the estimation of PDFs, $D_{KLx_i}^{p_f}\left(\hat{p}_f \mid p_f\right)$ may not be a good measure to discern the relative importance among those insignificant variables. Such sensitivity information is useful when the actual reliability of a failure mode does not meet the required level. By reducing a partial variation in P_1, such as 30%, the reliability of g_2 and g_{10} would reach a level very close to 1. On the other hand, little improvement on reliability will be expected when even totally eliminating the uncertainties in those insignificant random variables. For example, when fixing X_3 or X_8 at their mean values, the reliability of g_2 can only gain 0.005% improvement.

6. CONCLUSION

In this paper, we demonstrate that probabilistic sensitivity analysis (PSA) is a useful tool in design under uncertainty by providing valuable information about the impact of variables on the probabilistic characteristics of a response. Based on the range of interest of a design performance (i.e., robust design or reliability based design), PSA could be classified into two categories: the global response probabilistic sensitivity analysis (GRPSA) and the regional response probabilistic sensitivity analysis (RRPSA). Existing PSA methods such as the variance-based methods can only be used for GRPSA. To overcome this difficulty, we propose in this work a modification of Kullback-Leibler relative entropy based method that can be used for both GRPSA and RRPSA.

Demonstrated by three numerical examples as well as two engineering design problems, we show that the proposed approach is effective and flexible to be used under various design scenarios and at different design stages. By comparing our proposed entropy based method with the Sobol' method for GRPSA, we observe that our proposed method provides better quantification of the impact of a variable due to the incorporation of the PDF of response variable, while the Sobol' method captures only the impact on the response variance (second moment). In many cases, the rankings obtained from these two methods could be similar, however, for highly skewed and heavily-tailed distributions, the rankings will be different and our proposed entropy method provides a better measure of ranking in such cases.

Our study also shows that the proposed entropy based method can be easily extended for RRPSA. The two alternative formulations for RRPSA are both valid, while the formulation that treats reliability satisfaction as a discrete event might be more sensitive to numerical errors when dealing with very small values of failure rate. To alleviate this problem, parametric distribution such as pareto distribution may be employed to separately fit the tail distribution. This approach is subject to further research. Our examples also demonstrate the potential of RRPSA as an effective tool to quantify the impact of uncertainty reduction among the variables for reliability improvement.

One limitation of the relative entropy based PSA method is that they could only show a ranking of relative importance, but the value itself does not have absolute physical meaning. Considering the computation demand of PSA, future research will be directed towards improving the computational efficiency of using the proposed method.

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