Local Approximation of the Efficient Frontier in Robust Design

Jinhuan Zhang, Graduate Assistant

Department of Mechanical Engineering

Clemson University

Margaret M. Wiecek, Associate Professor

Department of Mathematical Sciences

Clemson University

Wei Chen*, Assistant Professor

Department of Mechanical Engineering

University of Illinois at Chicago

***Corresponding Author**
Dr. Wei Chen
Mechanical Engineering (M/C 251)
842 W. Taylor St.
University of Illinois at Chicago
Chicago IL 60607-7022

Phone: (312) 996-6072
Fax: (312) 413-0447
e-mail: weichen1@uic.edu

ABSTRACT

The multiple quality aspects of robust design have brought more and more attention in the advancement of robust design methods. Neither the Taguchi's signal-to-noise ratio nor the weighted-sum method is adequate in addressing designer's preference in making tradeoffs between the mean and variance attributes. An interactive multiobjective robust design procedure that follows upon the developments on relating utility function optimization to a multiobjective programming method has been proposed by the authors. This paper is an extension of our previous work on this topic. It presents a formal procedure for deriving a quadratic utility function at a candidate solution as an approximation of the efficient frontier to explore alternative robust design solutions. The proposed procedure is investigated at different locations of candidate solutions, with different ranges of interest, and for efficient frontiers with both convex and nonconvex behaviors. This quadratic utility function provides a decision maker with new information regarding how to choose a most preferred Pareto solution. As an integral part of the interactive robust design procedure, the proposed method assists designers in adjusting the preference structure and exploring alternative efficient robust design solutions. It eliminates the needs of solving the original bi-objective optimization problem repeatedly using new preference structures, which is often a computationally expensive task for problems in a complex domain. Though demonstrated for robust design problems, the principle is also applicable to any bi-objective optimization problems.

Keywords: Robust Design, Bi-Objective Optimization, Efficient Solutions, Utility Function, Decision Analysis

NOMENCLATURE

BORD	Bi-objective Robust Design
$f_i(x)$	i-th Objective Function
$F(x)$	Vector of Objective Functions
$\overline{F}$	Candidate Efficient Solution
$g_j(x)$	j-th Constraint Function
$g_j'(x)$	j-th Constraint Function in Robust Design
k_j	Penalty Factor
$q(F)$	Quadratic Utility Function of F
R	Range of Interest for Approximation
S/N	Signal/Noise
w	Vector Weights
WS	Weighted-sum
WSP(w)	Weighted-sum Problem
x	Vector of Design Variables
X	Design Space
x_L	Lower Bound for Design Variables
x_U	Upper Bound for Design Variables
x^*	Optimal Solution for Design Variables
$\overline{x}$	Candidate Solution Point in Design Space
Δx	Deviations (or Tolerance) of the Design Variables
α	Coefficient of QUF
β^*	Optimal Solution of β-problem
u^*	Utopia Point in CP
μ_f	Mean of the Objective Function f(x)
μ_f^*	Optimal Value of the Mean
σ_f	Standard Deviation of the Objective Function f(x)
σ_f^*	Optimal Value of the Standard Deviation

1. INTRODUCTION

Robust design, originally proposed by G. Taguchi (Taguchi, 1993), is a design method for improving the quality of a product through minimizing the effect of the causes of variation without eliminating the causes (Phadke, 1989). While the advancement of robust design methods in the statistical community has focused on the improvement of the efficiency of Taguchi's experimentation strategy and the modification of the signal-to-noise ratio as the robust design criterion (Box, 1988; Nair, 1992), the developments in the design community have produced nonlinear programming based robust design methods that can be used in a variety of applications (Otto and Antonsson, 1993; Parkinson et al., 1993; Sundaresan et al., 1993; Cagan and Williams, 1993; Chen et al.1996; Su and Renaud, 1997), including probabilistic-based robust optimization (Eggert and Mayne, 1993).

With the introduction of the nonlinear programming framework to robust design, the issue that has brought more and more attention in recent developments is the modeling of the multiple quality aspects of robust design. It has been recognized that the robust design objective could be generalized into *two aspects*, namely, "optimizing the mean of performance" and "minimizing the variation of performance". Since the performance variation is often minimized at the cost of sacrificing the best performance, the tradeoff between the aforementioned two aspects cannot be avoided. For modeling designer's preference structure of the two robust design aspects, different methods have been proposed following different paradigms for multiobjective decision making. Iyer and Krishnamurty (1998) presented a preference-based metric for robust design using concepts from utility theory (von Neumann and Morgenstern, 1947; Keeney and Raifa, 1976; Hazelrigg, 1996; Thurston, 1991) to capture designer's intent and preference when making the tradeoffs between the mean and variance attributes. Under the notion of utility theory, the ultimate overall worth of a design is represented by a *single* multiattribute utility function which incorporates consideration of attributes that cannot be directly converted to a common metric. Other researchers employed the multicriteria optimization approach and treated robust design problems as bi-objective nonlinear programming problems (Das and Dennis, 1996; Mulvery, et al. 1995). The focus is to generate the complete efficient solution set (Pareto curve) to support decision making.

An interactive multiobjective robust design procedure was proposed by the authors to solve bi-objective robust design (BORD) problems under the notion that a reliable mathematical representation of the decision-maker's actual utility function may not always be available (Chen et. al 1998). The proposed robust design procedure follows upon the recent developments on relating utility function optimization to a multiobjective programming method. It is developed to allow a designer to exercise his/her preference structure of the multiple aspects of robust design and to explore alternative solutions in an iterative manner. The details of this procedure are provided in Section 2.1. As an overview, there are two major elements of the proposed approach. The first element is associated with the use of the Compromise Programming (CP) (Yu, 1973 and Zeleny, 1973) approach, i.e., the Tchebycheff method, in replacement of the conventional weighted sum (WS) method. The second element is the derivation of the quality utility at a candidate solution by means of a quadratic function in a certain sense equivalent to the weighted Tchebycheff metric. It was then proposed to further use this utility function to explore the efficient solutions in a neighborhood of the candidate solution. In Chen et al. 1998, the advantages of the CP approach over the WS method in locating the efficient multiobjective robust design solutions (Pareto points) (Steuer, 1986) are thoroughly illustrated both in principle and through the example problems. The derivation of a family of quadratic utility functions at the considered efficient solution is demonstrated, however, the methodology on how to choose the parameter α that yields the most accurate quadratic function supporting the efficient frontier at the considered efficient solution was not provided. Our aim in this paper is to provide an extension of our previous work on developing an interactive multiobjective robust design procedure. The focus is on providing theoretical foundations and developing a procedure of determining an appropriate quadratic utility function for a given size of neighborhood. It is also our interest to investigate the validity of employing quadratic utility functions for both convex and nonconvex portions of the efficient frontier. As an integral part of the interactive robust design procedure, the developments in this paper are essential to assist decision making in quality engineering and applications of robust design. The principles demonstrated are not solely restricted to robust design problems, but also valid for any bi-objective optimization applications.

2. TECHNOLOGICAL BASIS OF OUR APPROACH

In this section, the interactive robust design procedure is first reviewed as a whole. The technical details of the two major elements of the proposed approach are then explained. The research issues to be addressed in this paper are raised at the end.

2.1 The Interactive Robust Design Procedure

The proposed interactive robust design procedure is presented in Figure 1. The first step of the proposed procedure is to transform a conventional optimization problem into a robust design formulation. For an engineering design problem stated using the conventional optimization model in Eqn. (2.1):

$$
\begin{aligned}
&\text{minimize} && f(x) \\
&\text{subject to} && g_j(x) \le 0, \qquad j = 1, 2, \ldots, J \qquad (2.1)\\
& && x_L \le x \le x_U,
\end{aligned}
$$

where x, x_L, and x_U are vectors of design variables, their lower bounds and upper bounds, respectively; f(x) stands for the objective function and $g_j(x)$ is the j-th constraint function. The robust design model is stated as a bi-objective robust design (BORD) problem as the following:

$$
\begin{aligned}
&\text{minimize} && [\,\mu_f, \sigma_f\,] \\
&\text{subject to} && g_j(x) + k_j \sum_{i=1}^{n} \left| \frac{\partial g_j}{\partial x_i} \right| \Delta x_i \le 0, \ \ j=1, 2, \ldots, J \qquad (2.2)\\
& && x_L + \Delta x \le x \le x_U - \Delta x,
\end{aligned}
$$

where μ_f and σ_f are the mean and standard deviation of the objective function f(x), respectively. In Eqn. (2.2), to study the variation of constraints, the original constraints are modified by adding the penalty term to each of them, where k_j (j=1, …,J) are penalty factors to be determined by the designer. The bounds of design variables are also modified to ensure feasibility under deviations.

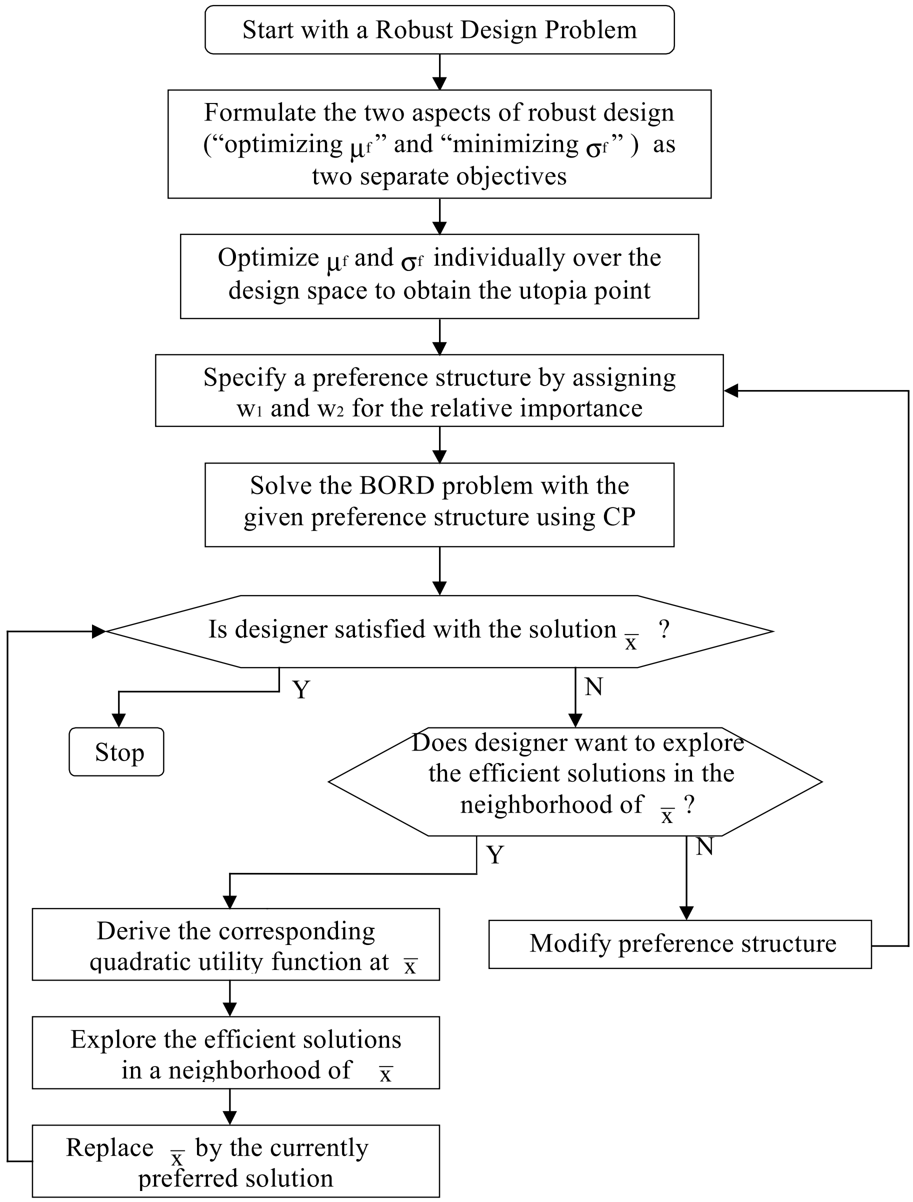

Figure 1 A Robust Design Procedure for Addressing Multiple Aspects of Robust Design (Chen et al. 1998)

The BORD is solved next using the CP approach for a given preference structure. Different from the conventional WS method, the basic idea in CP is to identify an ideal solution as a point (utopia point $[\mu_f^*, \sigma_f^*]$ in BORD) at which each attribute under consideration achieves its optimum value and seek a solution that is as close as possible to the ideal point. For a given preference structure, assigned by weights w_1 and w_2 representing the relative importance of the two objectives, the BORD problem is formulated using the following form of the CP, the weighted-Tchebycheff method:

$$
\begin{aligned}
&\text{minimize} && \beta \\
&\text{subject to} && w_1\left(\frac{\mu_f}{\mu_f^*} - 1.0 + \delta_1\right) \le \beta \\
& && w_2\left(\frac{\sigma_f}{\sigma_f^*} - 1.0 + \delta_2\right) \le \beta \\
& && g_j(x) + k_j \sum_{i=1}^{n} \left|\frac{\partial g_j}{\partial x_i}\right| \Delta x_i \le 0, \quad j=1, 2, \ldots, J \qquad (2.3) \\
& && x_L + \Delta x \le x \le x_U - \Delta x.
\end{aligned}
$$

where w_1, $w_2 > 0$, $w_1 + w_2 = 1$, and $\delta_i \geq 0$, $i = 1, 2$ in order to guarantee that the above formulation yields all the Pareto solutions. The above CP model can be used for generating all the Pareto solutions of the BORD problem. At this point, the designer is asked whether she/he is satisfied with the candidate solution, denoted as $\bar{x}$. If the answer is yes, the robust design procedure stops. Otherwise, the designer will be given the option of either exploring the set of efficient solutions in a neighborhood of the $\bar{x}$ or modifying the preference structure to solve again the BORD using CP. When the designer is interested in exploring the efficient solutions in a neighborhood of the $\bar{x}$, a technique used to derive the corresponding quadratic utility function at this point is used. This function effectively illustrates the robust design (quality) utility in a small neighborhood of $\bar{x}$, and could be used as an approximation of the efficient frontier to explore alternative solutions in the objective space. The process continues until a satisfactory solution is reached.

2.2 CP versus WS

The CP approach, i.e., the Tchebycheff method, is used in replacement of the conventional weighted-sum (WS) method to determine the robust design solution. A closer look at the drawbacks of minimizing weighted sums of objectives in multicriteria optimization is provided by Das and Dennis (1997). Though the weights representing relative importance are used as the preference structure when applying CP, it has been mathematically proven that CP is superior to the weighted-sum (WS) method in locating the efficient solutions, or the so called Pareto points (Steuer, 1986). Due to Geoffrion (1968), for every Pareto point of a convex multiobjective optimization problem there exists a (nonzero) vector weight $w \geq 0$ such that this Pareto point is an optimal solution of the WSP(w). *However, not every Pareto solution of a general (nonconvex) problem can be found by solving the corresponding WSP(w).* Moreover, it is found that for convex problems, an even spread of weights w may not produce an even spread of points in the efficient set. On the contrary, Bowman (1976) shows that for every Pareto solution of a general problem there exists a positive vector of weights so that the corresponding weighted-Tchebycheff problem is solved by this Pareto point.

2.3 Relating Utility Function to Efficient Frontier

The quadratic utility function is derived at a candidate solution as an approximation of the efficient frontier to explore the alternative robust design solutions in a neighborhood of the candidate solution. This eliminates the need of solving the CP problem repeatedly using new preference structures, which is often a computationally expensive task for problems in a complex domain.

The theoretical foundation of deriving the quadratic utility function is based on the work of Tind and Wiecek (1997) that shows that under certain conditions, the Pareto solution found by means of the Tchebycheff approach for a given utopia point and a given vector of weights can also be generated through the minimization of a quadratic function of the original objective functions. For a bi-objective optimization problem, the quadratic problem is formed as

$$\begin{aligned} &\text{minimize} \qquad q(F(x)) = F(x)^T QF(x) + p^T F(x) + c \qquad (2.4)\\ &\text{subject to} \qquad x \in X, \end{aligned}$$

where F(x) is a vector composed of the two original objective functions of the bi-objective optimization problem, Q is a symmetric 2 x 2 matrix of the form

$$Q = (1/2)\,\alpha \begin{bmatrix} w_1^2 & -w_1 w_2 \\ -w_1 w_2 & w_2^2 \end{bmatrix}, \qquad (2.5)$$

p is a vector, $p \in R^2$:

$$p = \begin{bmatrix} -\alpha w_1^2 u_1^* + \alpha w_1 w_2 u_2^* + w_1 y_1 \\ -\alpha w_2^2 u_2^* + \alpha w_1 w_2 u_1^* + w_2 y_2 \end{bmatrix}, \qquad (2.6)$$

and $$c = (u^*)^T Q\, u^* + w_1 y_1\, u_1^* + w_2 y_2\, u_2^*. \qquad (2.7)$$

The matrix Q, the vector p, and the scalar c involve new parameters: α and y; α is a positive scalar sufficiently large and $y \in R^2$ is entirely determined by the weights:

$$(y_1, y_2) = \left(\frac{w_2^2}{w_1^2 + w_2^2}, \frac{w_1^2}{w_1^2 + w_2^2}\right). \tag{2.8}$$

It can be shown that the matrix Q is positive semi-definite so that the quadratic problem involves the minimization of a convex quadratic function of the objective functions over the image of the design space in the objective space. Although the decision maker's utility is typically maximized, in this approach it is minimized due to the fact that the original bi-objective problem involves minimization. As a result, the decision maker's utility represented by the weighted Tchebycheff metric achieves a new representation by means of this convex function**.** The weighted-Tchebycheff problem is easy to solve while the quadratic problem is computationally complex. Most importantly, this quadratic function provides a decision maker with new information regarding how to choose the most preferred Pareto solution.

2.4 Issues To-Be-Addressed

It can be noted from the discussion in Section 2.3 that the resulting quadratic function in Eqn. (2.4) at the considered efficient solution stands for a family of the quadratic functions parametrized by α, a positive scalar that determines the steepness (or flatness) of the family functions in a neighborhood of $\overline{F} = F(\overline{x})$, the image of the design $\overline{x}$ in the objective space (see Figure 2).

According to Tind and Wiecek's theory (1997), the parameter α has to be large enough for the weighted-Tchebycheff problem and the quadratic problem to be related. This relationship is based upon the quadratic Lagrangian duality established between the two problems which guarantees that only for a sufficiently large α, the Pareto solution found by solving the weighted-Tchebycheff problem is also yielded when solving the quadratic problem. Therefore there exists a minimum value α_{min} such that for every $\alpha \geq \alpha_{min}$ the existence of a suitable quadratic function is guaranteed. The value α_{min} yields the flattest quadratic function supporting the efficient frontier at the considered efficient solution and therefore being the best one to inform the designer about the local tradeoffs.

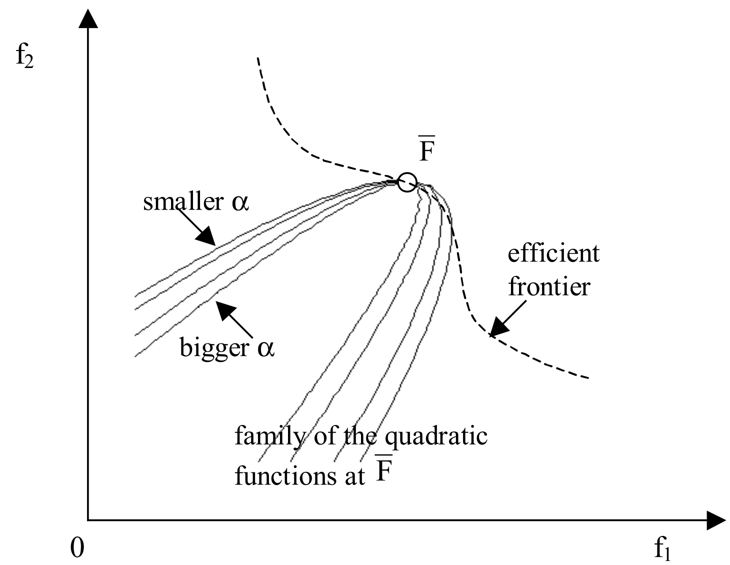

Figure 2 The Family of the Quadratic Utility Function at $\overline{F}$

As Tind and Wiecek (1997) did not provide theoretical foundations on how to choose α_{min}, it is our interest in this paper to develop a methodology that supports the derivation of the parameter α that results in the quadratic utility function with the best prediction accuracy of the efficient frontier. The challenge we are facing is: *without knowing the shape of the efficient frontier beforehand, how should we determine the value of α so that the shape of the quadratic function will be close to that of the efficient frontier?* In connection with the nature of our proposed interactive robust design procedure, two other research issues are brought into consideration under this development. The first is the issue of how to incorporate the size of the neighborhood (range of interest) into the procedure. The second is the question of whether the proposed method will be applicable to deriving the quadratic utility function at a candidate solution that may belong to either a convex or nonconvex portion of the efficient frontier.

3. A GENERAL PROCEDURE TO DETERMINE THE QUADRATIC UTILITY FUNCTION

According to the introduction in Section 2.2, the shape (flatness or steepness) and the location of the quadratic utility function is determined by the scalar α and the candidate efficient solution point $\overline{F}$, respectively. In this section, a general procedure for determining a proper quadratic utility function and two specific methods for getting α are described in detail. *Our proposed approach is based on the assumption that, within the range of interest, and within the family of the quadratic utility functions derived at the candidate solution, the best function should be the one that minimizes the difference between the true efficient solutions and the predicted efficient solutions derived from the quadratic utility.* Minimizing this difference is, to some extent, equivalent to minimizing the difference between the quadratic utility function values at these two groups of solutions. Mathematically, this means we can compute α such that $\varphi(\alpha)$ in Eqn. (3.1) attains its minimum,

$$\varphi(\alpha) = \sum_{k=1}^{K} (q(\overline{F}^k) - q(F^k))^2 = \sum_{k=1}^{K} (q(\overline{F}) - q(F^k))^2 , \qquad (3.1)$$

where $\overline{F}^k$ stands for the k-th predicted efficient solution and F^k (k=1, 2, ..., K) represents the k-th true efficient solution. Since the quadratic utility function value for all the predicted efficient solutions is the same as that at the candidate solution $\overline{F}$, the equation is further simplified as shown. Because $\varphi(\alpha)$ is a real function with only one variable α, we can solve for α from $\partial\varphi(\alpha)/\partial\alpha=0$. We have

$$\alpha = \frac{\sum_{k=1}^{K}[d_2(\overline{F}) - d_2(F^k)][d_1(\overline{F}) - d_1(F^k)]}{\sum_{k=1}^{K}[d_1(\overline{F}) - d_1(F^k)]^2}, \tag{3.2}$$

where

$$d_1(F) = 0.5(w_1 f_1 - w_2 f_2)^2 + w_1 w_2 u_2^* f_1 + w_1 w_2 u_1^* f_2 - w_1^2 u_1^* f_1 - w_2^2 u_2^* f_2 \tag{3.3}$$

$$d_2(F) = w_1 y_1 f_1 + w_2 y_2 f_2. \tag{3.4}$$

To calculate the left hand sides of Eqns. (3.3) and (3.4), we need to have several (K) true efficient solutions F^k $[f_1^k, f_2^k]$ within a neighborhood of the candidate solution according to the range of our interest in the objective space. The following question arises: how should we obtain these efficient solutions without solving the CP models beforehand? This question can be answered by solving a set of ε-constraint problems (Haimes and Chankong, 1983). Within the range of interest, for a given f_1^k value, the ε-constraint model can be used to generate the corresponding f_2^k so that F^k $[f_1^k, f_2^k]$ is an efficient solution. The same principle is also applicable when solving for f_1^k given an f_2^k value. The concept of the ε-constraint method is illustrated in Figure 3, while its mathematical formulation is provided in Eqn. (3.5). As shown in Figure 3, point A is the candidate solution obtained from the CP approach for a given preference structure. To derive the best quadratic utility function at this point, it is decided to choose points B and C as the two other true efficient solutions within the range of interest defined in the direction of the f_1 axis. For the given f_1^k values at points B and C, the corresponding f_2^k values can be found by setting ε_k in Eqn. (3.5) as ε_1 and ε_2, respectively.

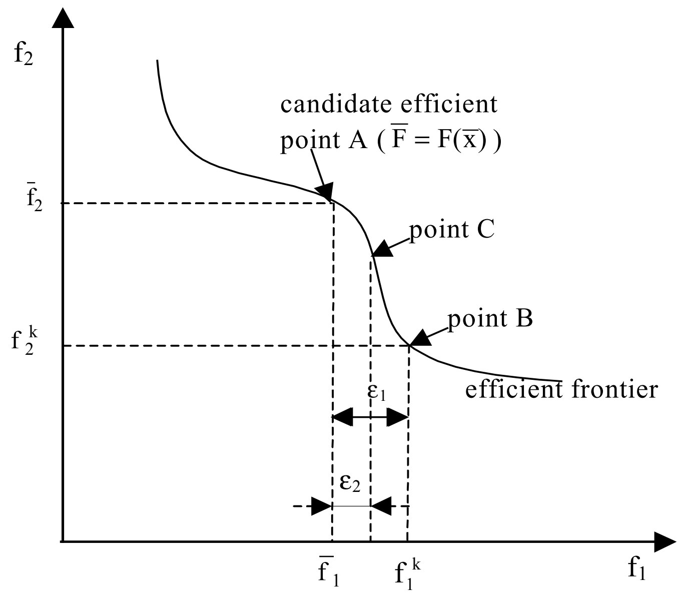

Figure 3 The Interpretation of the ε-constraint Method

$$
\begin{aligned}
&\text{minimize} && f_2(x) \\
&\text{subject to} && f_1(x) \le \bar{f}_1 + \varepsilon_k \qquad (3.5) \\
& && x \in X,
\end{aligned}
$$

where X denotes the design space that is formed by both the design constraints and the bounds on design variables x. The ε-constraint problem yields an efficient solution in two cases. If x^* is a unique solution of the problem and $f_1(x^*) = \bar{f}_1 + \varepsilon_k$, then the point $(f_1(x^*), f_2(x^*))$ is located on the efficient frontier. Alternatively, if we cannot ensure that x^* is unique we have to solve two ε-constraint problems, the one above and the other in which the objective f_1 is minimized while the objective f_2 generates the ε-constraint.

When the same x^* is an optimal solution of both problems and at the optimality of each problem the ε -constraint is satisfied as an equality, then the point $(f_1(x^*), f_2(x^*))$ is also located on the efficient frontier.

As a summary, the general procedure of selecting the parameter α for the quadratic utility function can be presented using the following figure. It is shown to be an iterative procedure to adjust the range of interest in case the range initially specified is too big to meet the error limit.

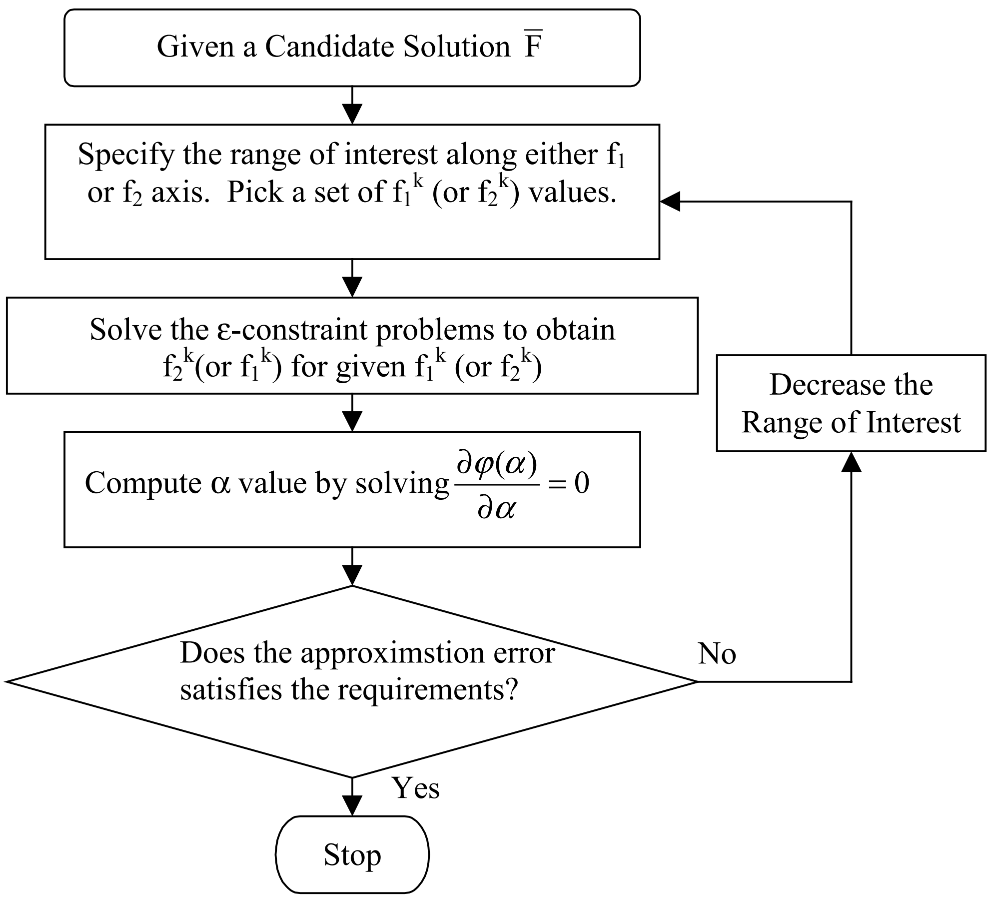

Figure 4. A General Procedure for Determining Parameter α

Theoretically the more true efficient solutions chosen within the range of interest, i.e., the larger the K, the better the accuracy of the quadratic utility function in approximating the efficient frontier. However, obtaining one efficient solution requires solving at least one ε-constraint optimization problem. The computational cost would be very high if too many efficient solutions were used. In this work, two specific methods, namely, the two-point method and the three-point method are introduced.

Two-Point Method

Two is the minimum number of points required to uniquely determine the parameter α in the quadratic utility function. Besides the candidate efficient solution $\overline{F}$, another efficient solution F^1 is obtained at the boundary of the range of interest by solving the ε-constraint problem. When K=1, Eqn (3.2) becomes:

$$\alpha = \frac{d_2(\overline{F}) - d_2(F^1)}{d_1(\overline{F}) - d_1(F^1)}, \tag{3.6}$$

Three-Point Method

One can imagine that, when F^1 is close to $\overline{F}$, α obtained by the two-point method should be satisfying. However, this may not be true when the range of interest is large. To improve the approximation accuracy, the three-point method is recommended instead of the two-point method. In addition to the candidate solution $\overline{F}$, two other efficient solutions F^1 and F^2 are obtained in the middle and at the boundary of the range of interest. The value of α is computed as:

$$\alpha == \frac{\sum_{k=1}^{2}[d_2(\overline{F}) - d_2(F^k)][d_1(\overline{F}) - d_1(F^k)]}{\sum_{k=1}^{2}[d_1(\overline{F}) - d_1(F^k)]^2}. \tag{3.7}$$

Our proposed procedure is illustrated and verified using an example problem.

4. EXAMPLE PROBLEM

4.1 Problem Formulations

The following mathematical problem, the same as the one used in Chen et al. 1998, is utilized to illustrate the tangible effects of our proposed approach.

$$\begin{aligned}
\text{minimize} \quad & f(x) = (x_1 - 4.0)^3 + (x_1 - 3.0)^4 + (x_2 - 5.0)^2 + 10.0 \qquad (4.1) \\
\text{subject to} \quad & g(x) = -x_1 - x_2 + 6.45 \leq 0 \\
& 1 \leq x_1 \leq 10 \\
& 1 \leq x_2 \leq 10 .
\end{aligned}$$

The optimal solution of the above problem is located at the point x = (1.21280,5.23742), with f(x) = -1.39378. The BORD for the mathematical problem is formulated as follows:

$$\begin{aligned}
\text{minimize} \quad & \left[\frac{\mu_f}{\mu_f^*}, \frac{\sigma_f}{\sigma_f^*}\right] \\
\text{subject to} \quad & g_1^{'}(x) = -x_1 - x_2 + 6.45 + 2k\Delta x \leq 0 \qquad (4.2) \\
& 1+\Delta x \leq x_1 \leq 10-\Delta x, \\
& 1+\Delta x \leq x_2 \leq 10-\Delta x,
\end{aligned}$$

where the mean and the standard deviation functions can be derived approximately using first-order Taylor expansion by considering the standard deviations of both x as $\Delta x/3$:

$$\mu_f(x) \cong (x_1 - 4.0)^3 + (x_1 - 3.0)^4 + (x_2 - 5.0)^2 + 10.0 \tag{4.3}$$

$$\sigma_f(x) \cong \frac{\Delta x}{3}\sqrt{(3.0(x_1 - 4.0)^2 + 4.0(x_1 - 3.0)^3)^2 + (2.0(x_2 - 5.0))^2}\,. \tag{4.4}$$

When the size of variation is considered as Δx=1.0 and the penalty factor k is taken as 1.0, the ideal solution is obtained as (μ_f^*, σ_f^*) = (5.10464, 0.416796) where $\mu_f^* = \mu_f(x_{\mu_f}^*)$ and $x_{\mu_f}^*$=(2.00000,6.45074), and $\sigma_f^* = \sigma_f(x_{\sigma_f}^*)$ and $x_{\sigma_f}^*$=(3.50559, 4.99187). To solve the BORD problem using the CP approach based on the formulation in Eqn. (2.3), we choose $\delta_1 = \delta_2$=1.0 and the utopia point becomes u^*=(0.0,0.0).

In Chen et al., 1998, a comparison is made between the solutions obtained from the WS method and the CP method for the above BORD problem. As illustrated in Figures 5 and 6, for eighteen evenly distributed combinations of w_1 and w_2 considered, the results obtained from the WS method and the CP method are distinctively different for some of the weight settings in the objective space. This example clearly illustrates that the CP method can generate the complete efficient set which is shown to have a segment of nonconvex curve as illustrated in Figure 6, while the WS method misses a large portion of the efficient frontier.

4.2 The Derivation of the Quadratic Utility Function (QUF)

We now explain how, for a given candidate solution obtained from the CP approach, the procedure proposed in this paper can be used to derive the quadratic utility function that best approximates the efficient robust design solutions in a neighborhood of the

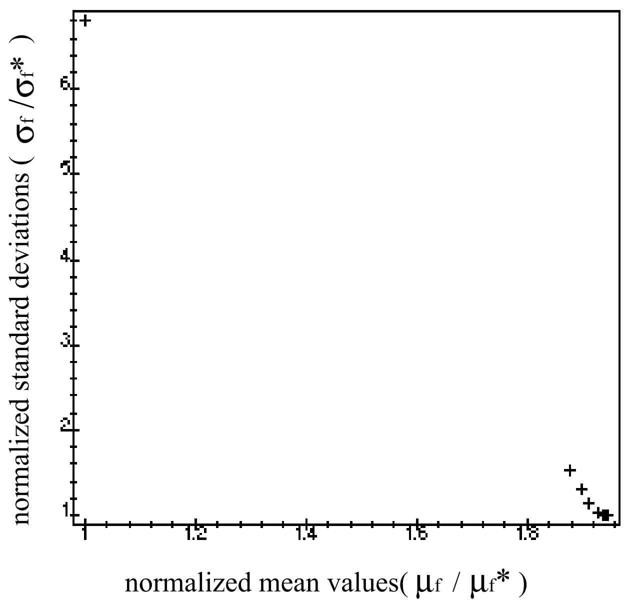

Figure 5 Efficient Solutions Using the WS Method (Chen et al. 1998)

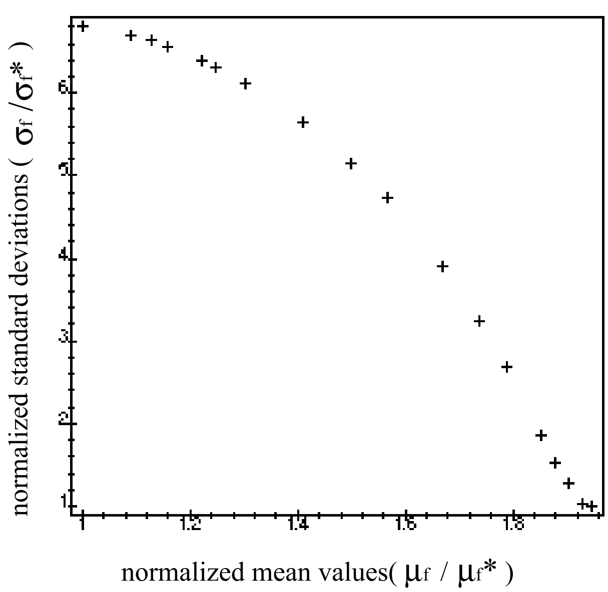

Figure 6 Efficient Solutions Using the CP Method (Chen et al. 1998)

candidate solution. Based on Eqns. (2.4)–(2.8), for the mathematical example problem, the quadratic function has the following form:

$$q(F) = 0.5\alpha w_1^2[f_1 - (w_2/w_1)f_2]^2 + p_1f_1 + p_2f_2\,. \qquad (4.5)$$

For a given preference structure, say w_1=0.90 and w_2=0.10, Eqn. (4.5) becomes

$$q(F) = 0.5 * \alpha * 0.90^2[f_1 - (0.10/0.90)f_2]^2 + 0.01098f_1 + 0.09878f_2 \qquad (4.6)$$

The above equation stands for a family of the quadratic functions parametrized by α. The proposed method introduced in Section 3 is used here to derive the most appropriate α. Following the procedure shown in Figure 4, the size of the neighborhood is determined by the range of interest along the f_1 axis. Considering the candidate solution as the left end of this range and the size of the range being R = 0.8, the range of f_1 is [0.999689, 1.799689]. The three-point method is applied here to derive α. In addition to the candidate solution, the other two points are at f_1= 1.399689 and f_1= 1.799689. For these two given values of f_1, the ε-constraint formulation in Eqn.(3.5) is used to yield the corresponding values of f_2, which are obtained as f_2 = 5.69783 and f_2 = 2.54514, respectively. Given the three efficient solutions including the candidate solution, α is obtained as 0.454607 based on Eqns. (3.3), (3.4) and (3.7,) and the quadratic function becomes:

$$q(F)=0.5 *0.454607* (0.90f_1-0.10f_2)^2+0.010976f_1+0.098781f_2\,, \qquad (4.7)$$

and at $\overline{F}$ and all the predicted efficient points the value q(F) = 0.694131.

The quadratic utility function in Eqn. (4.7) is graphically illustrated in Figure 7. It can be noted that this quadratic function serves as a good approximation of the efficient frontier within the range of interest. Table 1 compares the predicted values of f_2 based on the quadratic utility function and the true values of f_2 at some efficient solutions for a set of given f_1 values. The true values are takes from the results of CP method in our previous study (Chen et al., 1998). Among the points listed, the percent errors are all less than 5%. It is noted that the error is zero for $f_1 = 0.999689$. This is because the quadratic utility is derived using this point as one of the three points of interest.

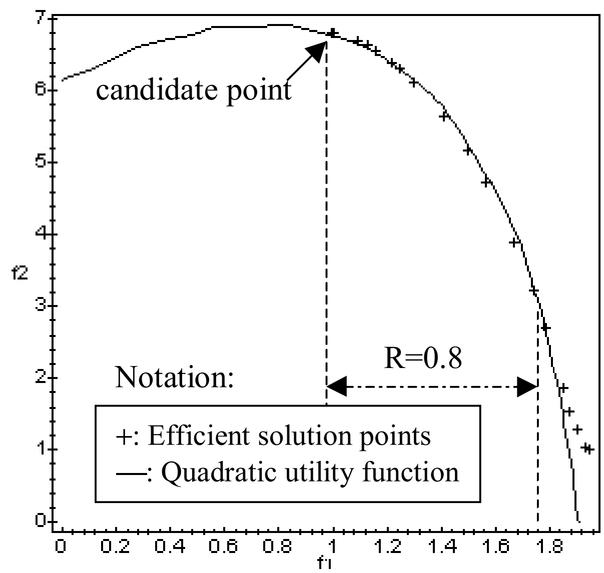

Figure 7 QUF Approximation Derived at [0.99969, 6.80552]

The results in Figure 7 and Table 1 illustrate that the quadratic utility function derived using the proposed procedure at the specified candidate solution is very accurate for approximating the efficient frontier.

4.3 Accuracy Under Different Schemes

Further studies are conducted to verify the accuracy of the quadratic utility functions derived at different candidate solutions and with different ranges of interest. Table 2 lists the results of several representative schemes. The α values determined by the proposed procedure are listed in the fifth column and the approximation accuracy is presented through the corresponding figures. Scheme I is the same as the study presented in Section 4.2. The approximation situations of the other schemes are shown in Figure 8 to Figure 10.

Table 1 Estimation of Efficient Solutions Using the QUF (α = 0.454607)

Given f_1 values	True f_2 values	f_2 estimated by QUF	Percent Difference of f_2
1.78753	2.688700	2.71094	0.8273%
1.67096	3.899960	4.07604	4.5149%
1.49778	5.158980	5.33361	3.3849%
1.30102	6.126120	6.17877	0.8595%
1.22104	6.400090	6.40861	0.1332%
1.12615	6.639740	6.61718	0.3397%
0.999689	6.805520	6.80552	0.0000%

From Figure 7 to 10, we note that the approximation accuracy of the quadratic utility function is better when it is derived at the solution (0.9997,6.8055) than at the solution (1.1592, 6.5619). Even with a large range of interest (R = 0.8, Scheme I), the error is very small. When derived at the solution (1.1592,6.5659), the accuracy is sensitive to the range chosen.

Table 2 QUF Obtained for Different Schemes

Scheme	Weighting sets (w_1,w_2)	Candidate Efficient Points $(\bar{f}_1,\bar{f}_2)$	Range of interest (R for f_1)	α value in QUF	Figure
I	(0.90,0.10)	(0.9997,6.8055)	0.8	0.45461	Figure 7
II	(0.90,0.10)	(0.9997,6.8055)	0.6	0.47882	Figure 8
III	(0.85,0.15)	(1.1592,6.5659)	0.8	0.71414	Figure 9
IV	(0.85,0.15)	(1.1592,6.5659)	0.4	1.39705	Figure 10

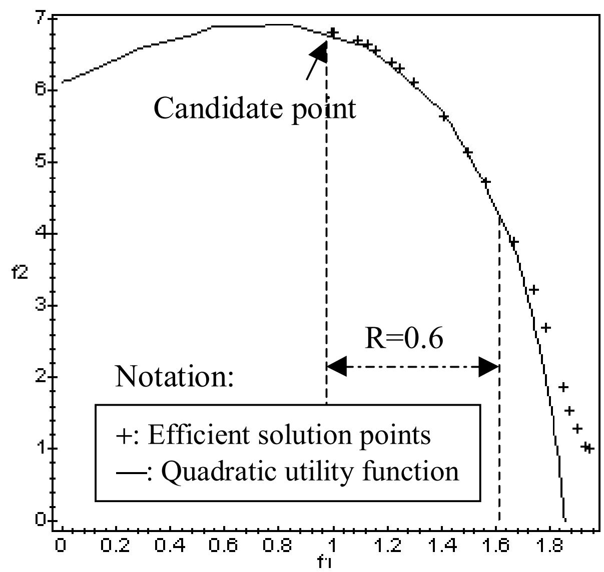

Figure 8 QUF Approximation (Scheme II)

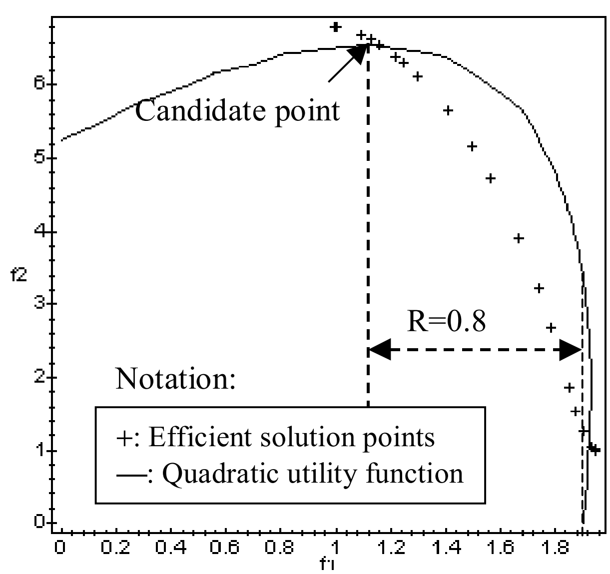

Figure 9 QUF Approximation (Scheme III)

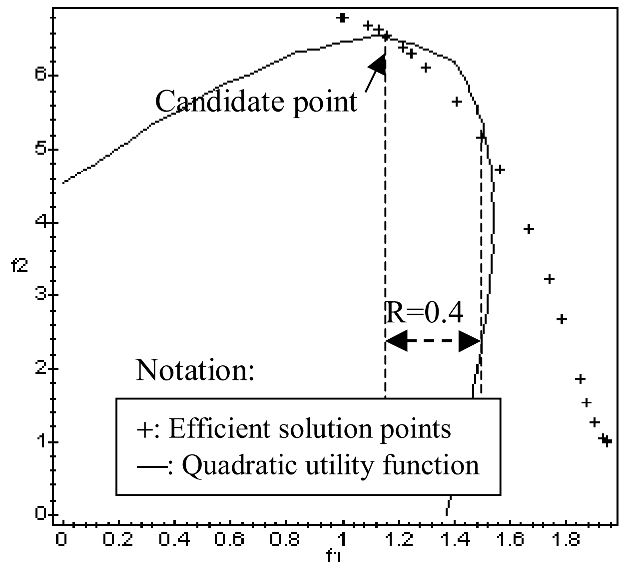

Figure 10 QUF Approximation (Scheme IV)

Tables 3 provides some details of the results for two different sizes of the range of interest (R = 0.8 and R = 0.4) at the candidate solution obtained for $w_1 = 0.85$ and $w_2 = 0.15$ (Schemes III & IV). From this table, we note that the approximation accuracy heavily depends on the approximation range. With R = 0.8 for f_1, the maximum error reaches 16.3%, while for R = 0.4 this error is controlled within 9.29%. The iterative characteristic of the proposed procedure can be used to lead the approximation to an acceptable level. Therefore at different candidate solutions, a suitable quadratic utility function can always be determined using the proposed three-point method for reasonable approximation ranges.

Table 3 Accuracy Over Different Ranges
(for $w_1 = 0.85$, $w_2 = 0.15$)

	approximation range(R=0.8)		*approximation range(R=0.4)*	
True f_2 value	f_2 (obtained by QUF)	error (%)	f_2 (obtained by QUF)	error (%)
6.126120	6.49982	6.1001	6.44992	5.2856
6.310140	6.53452	3.5558	6.51664	3.2725
6.400090	6.54839	2.3172	6.54049	2.1937
6.566890	6.56689	0.0000	6.56689	0.0000
1.24947	1.45341	16.3223	1.36557	9.2918
1.22104	1.39511	14.2555	1.32814	8.7714
1.15925	1.15925	0.00000	1.15925	0.00000

4.4 Other Issues

Our aim in this section is to study the validity of the proposed procedure for deriving quadratic utility functions to approximate both nonconvex and convex portions of the efficient frontier. Although the theoretical relationship between the weighted-Tchebycheff problem and the quadratic sum problem holds under certain conditions for a general multicriteria optimization problem (nonconvex or convex), the resulting approximation proposed in this paper works well for concave portions of the efficient frontier. For the candidate point that is located at a convex portion of the frontier, we propose that the same quadratic utility function as in Eqn. (2.4) be used but with α being the negative value of the one derived using Eqn.(3.2). In the objective space, this is equivalent to flipping the quadratic utility curve so that it supports the efficient frontier at the candidate solution from above rather than from below. Although the theoretical relationship between the weighted-Tchebycheff method and the weighted-quadratic problem, as presented in Section 2.3, does not hold for the new quadratic function, this function can still be used for approximation. Provided in Figures 11 and 12 are the comparisons of the quadratic utility functions when α takes a positive and a negative value. In Figure 11, α is derived for the candidate solution (1.5661,4.7222), which is the CP solution for w_1 = 0.75 and w_2 = 0.25, and the range of interest being R = 0.2. In Figure 12, α is derived for Scheme IV as shown in Table 2. In both cases, it is noted that the quadratic utility functions with positive α values approximate very well the concave portion of the efficient frontier at the right hand side of the candidate solution, while quadratic utility functions with negative α values serve as good estimations of the convex portion of the efficient frontier at the left hand side of the candidate solution.

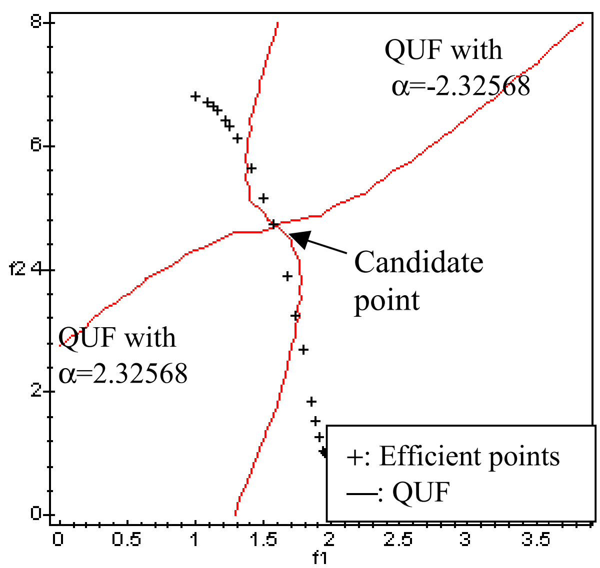

Figure 11 QUF with α= ± 2.32568

As the result of the above discussion, we may propose that a positive α be used to approximate concave portions of the efficient frontier while a negative α should be used for convex portions. However, the following question immediately arises: how can a designer recognize the shape of the efficient frontier (convex vs. concave) in a neighborhood of the candidate solution so that α can be properly chosen for the QUF. As the shape is unknown, one should derive both quadratic utility functions, the one with a positive α and the other with a negative α for the range of interest and then choose the one that yields a better approximation of the efficient frontier. The quality of the approximation may be measured by means of the approximation percentage error or another criterion introduced to the analysis. A detailed investigation of this issue, however, goes beyond the scope of this paper.

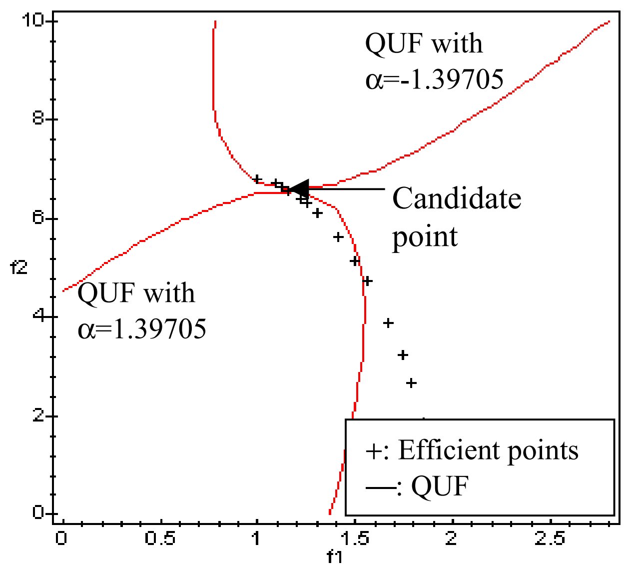

Figure 12 QUF with α= ± 1.39705

5. CLOSURE

In this paper, we present a formal procedure of deriving a quadratic utility function at a candidate solution as an approximation of the efficient frontier to explore alternative robust design solutions. At a specified candidate solution, the quality utility is represented by means of the quadratic function in a certain sense equivalent to the weighted Tchebycheff metric. For a given range of interest, the parameter α in the quality utility function is uniquely determined by minimizing the differences between some selected true efficient robust design solutions and the corresponding predicted efficient solutions. Through investigations at different locations of candidate solutions, with different ranges of interest, and for efficient frontiers with both convex and nonconvex behaviors, it has been found that a suitable quadratic utility function can always be determined using the proposed method within a reasonable range of interest. The sign of the parameter α can be decided upon so that the resulting quadratic function works well for concave as well as convex frontiers.

As an integral part of the interactive robust design procedure, the proposed method assists designers in adjusting the preference structure and exploring alternative efficient robust design solutions in a neighborhood of the candidate solution. It eliminates the need of solving the original bi-objective optimization problem repeatedly using new preference structures. This is practically significant for complex engineering design problems that are computationally expensive.

Once a favorable robust design solution is identified in the objective space, its value defines the target performance for both the mean and variance attributes. Goal programming techniques could then be employed to map the solution from the objective space to its pre-image in the design space.

In terms of the future work in this area of research, we plan to extend generating the quadratic utility function for robust design problems with more than two objectives. If successful, the proposed interactive procedure can be extended to any multiobjective optimization problems in a generic way.

ACKNOWLEDGMENT

The support from the NSF grant DMII 9624363 is gratefully acknowledged.

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