TOWARDS A BETTER UNDERSTANDING OF MODELING FEASIBILITY ROBUSTNESS IN ENGINEERING DESIGN

Xiaoping Du, Visiting Scholar

Wei Chen*, Assistant Professor

Integration & Design Engineering Laboratory (IDEL)
Department of Mechanical Engineering
University of Illinois at Chicago

***Corresponding Author**

Dr. Wei Chen
Mechanical Engineering (M/C 251)
842 W. Taylor St.
University of Illinois at Chicago
Chicago IL 60607-7022

Phone: (312) 996-6072
Fax: (312) 413-0447
e-mail: weichen1@uic.edu

ABSTRACT

In robust design, it is important not only to achieve robust design objectives but also to maintain the robustness of design feasibility under the effect of variations (or uncertainties). However, the evaluation of feasibility robustness is often a computationally intensive process. Simplified approaches in existing robust design applications may lead to either over-conservative or infeasible design solutions. In this paper, several feasibility-modeling techniques for robust optimization are examined. These methods are classified into two categories: methods that require probability and statistical analyses and methods that do not. Using illustrative examples, the effectiveness of each method is compared in terms of its efficiency and accuracy. Constructive recommendations are made to employ different techniques under different circumstances. Under the framework of probabilistic optimization, we propose to use a most probable point (*MPP*) based importance sampling method, a method rooted in the field of reliability analysis, for evaluating the feasibility robustness. The advantages of this approach are discussed. Though our discussions are centered on robust design, the principles presented are also applicable for general probabilistic optimization problems. The practical significance of this work also lies in the development of efficient feasibility evaluation methods that can support quality engineering practice, such as the Six Sigma approach that is being widely used in American industry.

Key words: Robust design, feasibility, most probable point, probabilistic optimization, quality engineering.

NOMENCLATURE

cdf	cumulative distribution function
F.	a *cdf*
f.	a *pdf*
g	a constraint function
I	indicator function
MPP	Most Probable Point
N	number of simulation
m	number of design parameters
n	number of design variables
P	probability
p	design parameter
P_0	desired probability of constraint satisfaction
pdf	probability density function
T	tolerance space
x	design variable
y	random design variables and parameters
u	basic variables in standard normal space
v	importance density function
W	corner space
w	weighting factor
F	*cdf* of standard normal distribution
b	safety index
e	error of probability using Monte Carlo simulation
m	mean value
s	standard deviation

1. INTRODUCTION

Deterministic optimization techniques have been successfully applied to a large number of engineering design problems. However, it is generally recognized that there always exist uncertainties in any engineering systems due to variations in design conditions, such as loading, material properties, physical dimensions of parts, and operating conditions. With the introduction of the integrated product and process development (IPPD) paradigm, manufacturing variations could be considered as another contributing source of uncertainty in the product design stage. Deterministic approaches do not consider the impact of such variations, and as a result, the design solution may be very sensitive to these variations. Moreover, deterministic optimization lacks the ability to achieve specified levels of constraint satisfaction (such as under reliability considerations). Therefore, a design based on the deterministic factor of safety may be infeasible or over-conservative.

Robust design, originally proposed by G. Taguchi (Taguchi, 1993), is a method for improving the quality of a product through minimizing the effect of the causes of variation without eliminating the causes (Phadke, 1989). With the introduction of the nonlinear programming framework to robust design (Otto and Antonsson, 1991; Parkinson et al., 1993; Sundaresan et al., 1993; Cagan and Williams, 1993; Eggert and Mayne, 1993; Chen et al.1996; Su and Renaud, 1997), both the robustness of design objectives as well as the robustness of design constraints can be considered. It is generally recognized that the robustness of a design objective can be achieved by simultaneously "optimizing the mean performance" and "minimizing the performance variance". Modeling the tradeoff between these two aspects has been widely studied in the literature (Sudaresan, et al., 1993; Bras and Mistree, 1995; Chen, et al., 1996; Iyer and Krishnamurty, 1998; Chen, et al., 1999). In general, objective robustness is an issue related to how to better model a designer's preference structure when making tradeoffs between the mean and variance attributes.

No matter what objective expression we use to achieve the robustness of product performance, it is even more critical to maintain the design feasibility under variations (uncertainties). For example, for a key structural component, satisfying strictly its strength constraint (or reliability) subject to random parameters is more important than achieving the robustness of the design objective, e.g., minimizing the weight. This raises the question: *how can we describe the design feasibility under the effect of variations to maintain feasibility robustness*? Moreover, as we will discuss later in detail, depending on the formulation, the evaluation of feasibility robustness could become a very complicated and time-consuming process. This leads to another question: *what kind of constraint model should we adopt to ensure the accuracy in evaluating levels of constraint satisfaction with an acceptable computational efficiency*?

Although alternative approaches, such as the probabilistic feasibility analysis (Eggert, et al., 1991), the moment matching method (including the use of Taylor expansion) (Parkinson, et al., 1993), the worst case analysis (Parkinson, et al., 1993; Sundaresan, et al., 1995), the method of corner space evaluation (Sundaresan, et al., 1993), and the variation patterns method (Yu and Ishii, 1998), have been proposed to model feasibility robustness, the effectiveness of each individual method in terms of its efficiency and accuracy is not clear. Koch et al. (1998) compared three methods (Taylor expansion, design of experiments (DOE)-based Monte Carlo simulation, and Taguchi's product array) for predicting performance variance. However, their study focused on only the evaluation of performance variance rather than the overall level of constraint satisfaction. Due to the lack of guidelines in the area of evaluating feasibility robustness, simplistic approaches such as the first order Taylor expansion and the worst case analysis are often used in existing applications.

Our aim in this paper is to conduct an in-depth analysis of the existing feasibility-modeling techniques in robust design and compare these methods using illustrative examples. We will show that, although some of these approaches are easy to use, they may lead to either over-conservative or infeasible design. Constructive recommendations are made to employ different techniques under different circumstances. To improve the accuracy and efficiency, we propose to use a most probable point (*MPP*) based importance sampling method, a method rooted in the field of reliability analysis, for evaluating the feasibility robustness. The advantages of this approach are discussed based on comparisons.

This paper is organized as follows. In Section 2, the existing methods for feasibility modeling in robust design are analyzed, and a most probable point (*MPP*) based importance sampling method is proposed. The use of this method for the evaluation of feasibility robustness is described. The comparison of different methods for feasibility evaluation is discussed in detail through illustrative examples in Section 3. Section 4 is the closure of this paper.

2. APPROACHES FOR MODELING FEASIBILITY ROBOUSTNESS

2.1 Objective Robustness and Feasibility Robustness

Consider an engineering design problem stated using the conventional optimization model in Eqn. (2.1):

$$\begin{aligned} &\text{minimize } F(\mathbf{x},\mathbf{p}) \\ &\text{subject to } \mathrm{g_j}(\mathbf{x},\mathbf{p}) \ge 0, \qquad \mathrm{j} = 1, 2, ..., \mathrm{J} \\ &\mathbf{x}_l \le \mathbf{x} \le \mathbf{x}_u , \end{aligned} \tag{2.1}$$

where $\mathbf{x} = [x_1, \cdots x_n]^T$ is a vector of design variables and $\mathbf{p} = [p_1, \cdots p_m]^T$ is a vector of design parameters whose values are fixed as a part of the problem specifications. In robust design, both design variables and design parameters could be the contributing sources of variations. Consequently, the system performance $F(\mathbf{x}, \mathbf{p})$ is a random function. Both its mean value $m_F(\mathbf{x},\mathbf{p})$ and variance $s_F^2(\mathbf{x},\mathbf{p})$ are expected to be minimized. The general form of the objective can be expressed as

$$\min \quad [m_F(\mathbf{x},\mathbf{p}), s_F(\mathbf{x},\mathbf{p})] \tag{2.2}$$

In a deterministic optimization as shown in Eqn. (2.1), those design points that satisfy all the constraint equations define the feasible region. This is a go or no-go problem and the limit-state of feasibility or unfeasibility is distinguished. In robust design, however, the problem needs to be converted into a consideration of the degree of feasibility between yes and no. According to Parkinson et al. (1993), a design is described to have "feasibility robustness", *if it can be characterized by a definable probability, set by designers, to remain feasible relative to the nominal constraint boundaries as it undergoes variations.* It is obvious that, compared with the deterministic feasible region, the size of the feasible region will be reduced under the robustness consideration. In addition, based on the above definition, we note that the degree of feasibility can be defined by the desired level of probability chosen by the decision maker. In the following sections, several existing feasibility modeling methods are analyzed, and the MPP based importance sample method is proposed. The existing methods are classified into two categories: methods that require probability and statistical analyses and those that do not require such analyses.

2.2 Existing Methods Requiring Probability and Statistical Analyses

The Probabilistic Feasibility Formulation

Under the definition of feasibility robustness in Section 2.1, a general probabilistic feasibility formulation can be expressed as follows:

$$P[g_j(\mathbf{x},\mathbf{p}) \geq 0] \geq P_{oj} \quad j = 1,\cdots,J\ , \tag{2.3}$$

where P_{oj} is the desired probability for satisfying constraint *j*. If the distributions of all the variables **x** and parameters **p** are known, the probability *P* in Eqn. (2.3) can be obtained accurately by the following integral:

$$P[g_j(\mathbf{x},\mathbf{p}) \geq 0] = \int_{g_j(\mathbf{x},\mathbf{p})\geq 0} f_{\mathbf{xp}}(\mathbf{x},\mathbf{p}) d\mathbf{x} d\mathbf{p}\ , \tag{2.4}$$

where $f_{\mathbf{xp}}(\mathbf{x},\mathbf{p})$ is the joint probability density function (*pdf*) of **x** and **p**.

Practically, it is very difficult or even impossible to get an analytical or numerical solution of the above equation because of the multi-dimensional integration and the complicated integral region. Only if the distribution of $g_j(\mathbf{x},\mathbf{p})$ is known, can the probability be simplified into the following one-dimensional integral:

$$P[g_j(\mathbf{x},\mathbf{p}) \geq 0] = \int_0^{\infty} f_{gj}(g_j)dg_j, \tag{2.5}$$

where $f_{gj}(g_j)$ is the *pdf* of $g_j(\mathbf{x},\mathbf{p})$. For several typical variable distributions (for example, Normal and Lognomal), when used for simple constraint functions and low-dimensional problems, the analytical expression of the probability can be derived (Eggert, 1991).

In the case that the analytical method is not applicable, simulation-based approaches, such as Monte Carlo simulation, are often used to obtain a more accurate estimation of the probability. The estimation of the probability is expressed as:

$$P[g_j(\mathbf{x},\mathbf{p}) \geq 0] = \int I[g_j(\mathbf{x},\mathbf{p})] f_{\mathbf{xp}}(\mathbf{x},\mathbf{p}) d\mathbf{x} d\mathbf{p} = \frac{1}{N}\sum_{i=1}^{N} I[g_j(\mathbf{x}_i,\mathbf{p}_i)], \tag{2.6}$$

where N is the simulation size, $\mathbf{x}_i$ and $\mathbf{p}_i$ are samples of **x** and **p**, and $I[\cdot]$ is an indicator function defined as

$$I[g_j(\mathbf{x},\mathbf{p})] = \begin{cases} 1 & if \quad g_j(\mathbf{x},\mathbf{p}) \geq 0 \\ 0 & \quad otherwise \end{cases} \tag{2.7}$$

Simulation methods are flexible for any types of distributions and any forms of constraint functions. Neglecting the algorithmic error caused by simulation, if a sufficient number of simulations are used, simulation methods often result in solutions with a high accuracy. Under $100(1-a)\%$ confidence, the error bound of the general Monte Carlo simulation is given by (Law and Kelton, 1982) as

$$e = z_{1-a/2}\sqrt{\frac{P(1-P)}{N}} \,, \tag{2.8a}$$

The percentage error is

$$\%e = 100 z_{1-a/2}\sqrt{\frac{P}{N(1-P)}} \,, \tag{2.8b}$$

where $z_{1-a/2}$ denotes the $(1-a/2)$ quantile of the standard normal distribution and P is the probability of constraint satisfaction. Monte Carlo Simulation has a remarkable feature that the error bound does not depend on the dimension. However, when the probability P is very high (approaching 1.0), simulation number N will increase greatly, and as a result, the computational burden may not be affordable. We propose to use a most probable point based importance sampling method to significantly improve the computational efficiency associated with probabilistic feasibility evaluations. The method will be introduced in Section 2.4 after the descriptions of the exiting methods.

The Moment Matching Formulation

To reduce the computational burden associated with the probabilistic feasibility evaluation, simplistic approaches are widely used in the literature. One of these approaches is the moment matching method (Parkinson et al. 1993). The title of this method comes from the fact that it uses the first and second moments (mean and variance) of statistical distributions. With this approach, if $g_j(\mathbf{x},\mathbf{p})$ is assumed to be normally distributed, the probability of the event $g_j(\mathbf{x},\mathbf{p}) \geq 0$ becomes:

$$P[g_j(\mathbf{x},\mathbf{p}) \geq 0] = \Phi(\frac{m_{gj}}{s_{gj}}) \,, \tag{2.9}$$

where $\Phi(\cdot)$ is the cumulative distribution function (*cdf*) of a standard normal distribution**,** and m_{gj} and s_{gj} are the mean value and the standard deviation of $g_j(\mathbf{x},\mathbf{p})$, respectively. The constraint can then be written as:

$$m_{gj} - k_j s_{gj} \geq 0, \tag{2.10}$$

where $k_j = \Phi^{-1}(P_{0j})$ and $\Phi^{-1}(\cdot)$ is the inverse function of the *cdf* of a standard normal distribution. For example, $k_j = 2$ stands for $P_{0j} = 0.9772$ and $k_j = 3$ means $P_{0j} = 0.9987$.

Several methods could be used to evaluate m_{gj} and s_{gj}. A simplistic approach is to use Taylor series approximations of the constraint function $g_j(\mathbf{x},\mathbf{p})$ at the mean values of **x** and **p**. The mean value and the variance of $g_j(x, p)$ are estimated as

$$m_{gj} = g_j(m_{\mathbf{x}}, m_{\mathbf{p}}) \tag{2.11}$$

$$s_{gj}^2 = \sum_{i=1}^{n} \left[\frac{\partial g_j}{\partial x_i}\right]^2_{m_{\mathbf{x}} m_{\mathbf{p}}} s_{xi}^2 + \sum_{i=1}^{m} \left[\frac{\partial g_j}{\partial p_i}\right]^2_{m_{\mathbf{x}} m_{\mathbf{p}}} s_{pi}^2 \tag{2.12}$$

2.3 Existing Methods Not Requiring Probability and Statistical Analyses

The Worst Case Analysis

Worst case analysis is another simplistic approach to the evaluation of feasibility robustness in robust design. It is applicable to general robust design problems including those in which the distributions of random variables are not given. The worst case analysis (Parkinson et al., 1993) assumes that all fluctuations may occur simultaneously in the worst possible combinations. The effect of variations on a constraint function is estimated from a first order Taylor's series as follows:

$$\Delta g_j(\mathbf{x},\mathbf{p}) = \sum_{i=1}^{n}\left|\frac{\partial g_j}{\partial x_i}\Delta x_i\right| + \sum_{i=1}^{m}\left|\frac{\partial g_j}{\partial p_i}\Delta p_i\right| \tag{2.13}$$

By subtracting $\Delta g_j(\mathbf{x},\mathbf{p})$ from $g_j(\mathbf{x},\mathbf{p})$ to maintain the feasibility, the constraint becomes:

$$g_j(m_{\mathbf{x}}, m_{\mathbf{p}}) - \left(\sum_{i=1}^{n}\left|\frac{\partial g_j}{\partial x_i}\Delta x_i\right|_{m_x m_p} + \sum_{i=1}^{m}\left|\frac{\partial g_j}{\partial p_i}\Delta p_i\right|_{m_x m_p}\right) \geq 0 \tag{2.14}$$

In most cases, the worst case analysis is conservative because it is unlikely that the worst cases of variable or parameter deviations will simultaneously occur. On the other hand, the estimation using Taylor expansion is not as accurate as identifying the extreme conditions such as the minimum and maximum of the performance within the given intervals of variations. However, due to its simplification, worst case analysis is used widely in robust optimization applications.

The Corner Space Evaluation

Following the similar idea of the worst case analysis, Sundaresan et al. (1995) presented the method of corner space evaluation. Identical to the worst case analysis, their method does not require descriptions of the distributions of random variables. However, the variations on design variables and design parameters are not transmitted into constraint functions as the way in the worst case analysis.

Assume that the design variables have nominal values $\mathbf{x}$ and a tolerance $\Delta\mathbf{x}$, and that the design parameters have nominal values $\mathbf{p}$ and a tolerance $\Delta\mathbf{p}$. The *tolerance space* (T) is defined as a set of points close to the target design point where each point represents a possible combination of design variables due to uncertainties in each variable:

$$T(\mathbf{x}_t, \mathbf{p}_t) = \{\mathbf{x}_t : |\mathbf{x}_t - \mathbf{x}| \le \Delta\mathbf{x}, \mathbf{p}_t : |\mathbf{p}_t - \mathbf{p}| \le \Delta\mathbf{p}\} \tag{2.15}$$

The corner space (*W*) consists only of corner vertices of a tolerance space:

$$W(\mathbf{x}_t, \mathbf{p}_t) = \{\mathbf{x}_t : |\mathbf{x}_t - \mathbf{x}| = \Delta\mathbf{x}, \mathbf{p}_t : |\mathbf{p}_t - \mathbf{p}| = \Delta\mathbf{p}\} \tag{2.16}$$

To maintain the design feasibility, the nominal value **x** should be inside the feasible region. This can be achieved by keeping the corner space always touching the original constraint (expressed by **x** and **p**) boundary. Fig. 1 shows the feasibility of a two-dimensional problem with this approach.

Insert Fig. 1 here.

With this approach, the constraint can be stated as:

$$Min\{g_j(\mathbf{x}, \mathbf{p}), \quad \forall \mathbf{x}, \mathbf{p} \quad \in W(\mathbf{x}, \mathbf{p})\} \ge 0 \quad j = 1, \cdots J\ . \tag{2.17}$$

If the distributions of variables of interest are known, the tolerance $\Delta\mathbf{x}$ and $\Delta\mathbf{p}$ can be determined by a prescribed confidence level. For example, for a normally distributed random variable, the tolerance can be chosen as three standard deviations under the confidence coefficient of 99.87%. Since this method does not require the calculation of the partial differential of the constraint function, it is very easy to use. However, the overall probability of constraint satisfaction is not evaluated as the result of this procedure even though the tolerance is under the consideration of confidence levels.

The Variation Patterns Formulation

In the same category of the corner space evaluation, Yu and Ishii (1998) presented an improved method named Manufacturing Variation Patterns (*MVP*) analysis based on the consideration that the manufacturing errors may be correlated with each other. Since the approach is not restricted to manufacturing related problems only, a general title "variation patterns formulation" is given. With their approach, *MVP(1-a)* denotes the space of possible variable combinations at the confidence coefficient of 1-α, where α indicates the probability of design variable distribution outside the variation pattern. The shape of the pattern is determined by the variable distributions, and the size of the pattern is determined by the confidence coefficient. For example, for the problem with two normally distributed dependant variables, the shape of the pattern is an ellipsoid as shown in Fig. 2. Under this concept, the constraint is formulated as:

$$g_j(\mathbf{x}, \mathbf{p}) \geq 0, \quad \forall \mathbf{x} \in MVP(1-a), \quad j = 1, \cdots J \tag{2.18}$$

Insert Fig. 2 here.

It is obvious that the process of searching for the robust design solution is quite complicated if the shape of the pattern is irregular.

2.4 The MPP based Importance Sampling Method For Probabilistic Feasibility Evaluation

From the preceding discussions, we note that the probabilistic feasibility formulation is the ideal method to describe the feasibility robustness, as it can ensure that the solution achieves an accurate level of constraint satisfaction. We will further demonstrate the usefulness of the probabilistic feasibility formulation through the example problems in Section 3. A part of our interests in this paper is *to develop an affordable probabilistic feasible evaluation technique, in replacement of the general Monte Carlo simulations, so* *that the probabilistic robust optimization framework can be used more widely in robust design practices.*

We propose to use a most probable point (*MPP*) based importance sampling method for evaluating the feasibility robustness. The *MPP* method was originally developed in the field of reliability analysis (Hasofer and Lind, 1974). It has received more and more attention in recent implementations of probabilistic optimization (Maglaras**,** et al., 1996). We extend the same principle to evaluating the feasibility robustness in robust design problems. For simplicity, we call both the random design variables and random parameters as basic random variables and use the vector $\mathbf{y} = [y_1, \cdots y_k]^T$ to denote them. The constraint can then be written as:

$$g(\mathbf{y}) \geq 0 . \tag{2.19}$$

We assume $y_i (i = 1, \cdots, k)$ are mutually independent and their probability density functions and cumulative distribution functions are denoted as $f_i(y_i)$ and $F_i(y_i)$, respectively. If $y_i (i = 1, \cdots, k)$ are correlated, an intermediate step is required to transform them into a random variable set which is uncorrelated (Melchers, 1999). Two steps are followed to calculate the probability of $P[g(\mathbf{y}) \geq 0]$. The first step is the search for the so-called most probable point (*MPP*), and then in the second step, probability is calculated by the importance sampling around the *MPP*.

The *MPP* method uses the properties of standard normal space. The basic random variables $\mathbf{y}$ are transformed into standard, uncorrelated, and normal variables $\mathbf{u} = [u_1, \cdots u_k]^T$. The transformation is given by the Rosenblatt transformation (Rosenblatt, 1952) as

$$u_i = \Phi^{-1}[F(y_i)] \tag{2.20}$$

Eqn. (2.19) can now be rewritten as

$$g(\mathbf{u}) \geq 0 \,. \tag{2.21}$$

In the transformed **u** space, the *MPP* is defined as the minimum distance point, which is the point in the **u** space that has the highest probability of producing the value of constraint function *g*(**u**) (Hasofer and Lind,1974, Wu, 1990) (see Fig. 3). The minimum distance β is called the safety factor.

Insert Fig. 3 here.

If the constraint function *g*(*Y*) is linear in terms of the normally distributed random variables **y**, the accurate probability of constraint satisfaction is given by the equation:

$$P[g(\mathbf{y}) \geq 0] = \Phi(\ b) \tag{2.22}$$

If the constraint function g(y) is nonlinear or random variables y are not normally distributed, a good approximation can still be obtained by the above equation, provided that the magnitude of the principal curvatures of the constraint surface in the **u** space at the *MPP* is not too large (Mitteau, 1999). If the constraint function is highly nonlinear, an alternative second-order approximation at the *MPP* can be used, which takes into account the curvature of the constraint surface around the *MPP* (Breitung, 1984, Tvedt, 1990). Different techniques can be used to search the *MPP*, such as the hypersphere method (Ticky, 1993), the directional cosines method (Ang and Tang, 1984), the advance mean value (*AMV*) (Wu, et al., 1990), the sampling-based *MPP* search (Wu, 1998), and the optimization method. After the *MPP* is obtained, samples are picked around the *MPP* to evaluate the probability of constraint satisfaction through importance sampling.

An importance-sampling density, $v_{\mathrm{y}}(\mathbf{y})$, is introduced into the Monte Carlo estimation equation (2.6) to obtain the probability of constraint satisfaction:

$$P[g(\mathbf{x},\mathbf{p}) \geq 0] = \int I[g(\mathbf{y})] \frac{f_{\mathrm{y}}(\mathbf{y})}{v_{\mathrm{y}}(\mathbf{y})} v_{\mathrm{y}}(\mathbf{y}) d\mathbf{y} \qquad (2.23)$$

A Monte Carlo algorithm to evaluate the integral in Eqn. (2.2) would be to sample a series of $\mathbf{y}_i$ from $v_{\mathrm{y}}(\mathbf{y})$ and to estimate the probability through

$$P[g(\mathbf{x},\mathbf{p}) \geq 0] = \frac{1}{N} \sum_{i=1}^{N} I[g(\mathbf{y}_i)] \frac{f_{\mathrm{y}}(\mathbf{y}_i)}{v_{\mathrm{y}}(\mathbf{y}_i)} \qquad (2.24)$$

The importance sampling can be implemented in the standard normal space **u,** and its density is chosen as the standard normal distribution with its mean value shifted to the *MPP* (Ang, et al., 1992). The concept is illustrated in Fig. 4. We can see that about half of the samples fall into either the unfeasible region or the feasible region. The evaluation efficiency can be significantly improved by this way. From Eqn. (2.8b), we note that for the same confidence level and the same percentage error, if the probability of constraint feasibility is set at 0.99, the number of general Monte Carlo simulation over that of the MPP based importance sampling will be 99: 1, and if the probability of constraint feasibility changes to 0.999, the ratio will increase dramatically to 999:1 (P is set at 50% in Eqn. 2.8b for importance sampling in this case).

Insert Fig. 4 here.

As the evaluation of probabilistic feasibility is a part of a robust design (optimization) process, we have taken into account the following measures to use this approach more effectively. When b obtained in the *MPP* search is far away from the desired probability (for instance, $b<1.0$ or $b>4.0$ for the desired probability 0.95), to reduce the computational effort, we may use $\Phi(b)$ as the approximation of the probability during those iterations of optimization. In this case, further sampling is not required. In the process of sampling, we recommend to determine the number of simulations by a prescribed error with a certain confidence level. The system will keep tracking the number of samples that fall into the feasible region and computing the simulation error due to randomness (Law and Kelton, 1982). If the error is less than the acceptable error defined by designers under a certain confidence level, the sampling process will stop, and the probability will be estimated. To keep the stability of convergence in an optimization process and to ensure the repeatability of solutions, we suggest using the same "seed" number to generate random variables.

3. A COMPARISON OF ALTERANTIVE TECHNIQUES

In this section, the existing feasibility-modeling techniques reviewed in Section 2 and the proposed *MPP* based importance sampling technique are compared using three examples. The first example only illustrates the differences in feasibility evaluation when using different approaches. The other two engineering design problems are used to illustrate the impact of different methods on both feasibility evaluation and the final robust design solution. Constructive recommendations are made to employ different techniques for modeling feasibility robustness under different circumstances.

3.1 A Mathematical Example

In this example, we consider a simple linear constraint to show how feasibility robustness affects the region of feasibility. Only two design variables x_1 and x_2 are involved, which are normally distributed and are represented as $x_1 \sim N(m_1, s_1)$ and

$x_2 \sim N(m_2, s_2)$, where $s_1 = c_1 m_1$, $c_1 = 0.2$ and $s_2 = 0.25$. The original constraint function is given as

$$g(\mathbf{x}) = x_1 - x_2 \tag{3.1}$$

The design variables in optimization are the mean values m_1 and m_2 of x_1 and x_2. In Figure 5, the obtained constraint feasibility evaluations are compared for using different methods. The constraint evaluated in the deterministic situation (a straight line) is also included as a reference. The desired probability of constraint satisfaction P_0 is set as 99.98%. It is noted that the probabilistic feasibility has resulted in a reduced feasible region compared to the deterministic constraint. The probabilistic feasibility can be derived directly for this example, and the proposed *MPP* method or Monte Carlo simulation is not needed. For this specific problem, the moment matching formulation is identical to the probabilistic feasibility formulation.

When using the worst case formulation, we set $\Delta x_1 = 3s_1$ and $\Delta x_2 = 3s_2$. From Fig. 3, we note that the use of worst case formulation is conservative over the majority of the design space. However, the problem becomes infeasible either near the origin or when m_1 is bigger than about 6.7, where the probability of constraint satisfaction is less than the expected probability. We can conclude from this example that though the worst case analysis is widely considered as a conservative approach for modeling feasibility robustness, we should use it with caution because the violation of constraints is still possible over certain design regions.

Insert Fig. 5 here.

When using the corner space formulation, we set $\Delta x_1 = 3s_1$ and $\Delta x_2 = 3s_2$, which indicate that the confidence coefficient is 99.87%. By keeping the rectangle with dimensions of $2\Delta x_1 \times 2\Delta x_2$ touching the deterministic constraint curve, we obtain the locus of the centroid of the rectangle that stands for the position of the constraint limit. For the special linear function in Eqn. (3.1), the constraint curve obtained by the corner space formulation is the same as the one from the worst case formulation.

3.2 Design of a Cantilever Beam

An engineering design problem is used to further illustrate the differences between the existing approaches and their impacts on final robust design solutions. The cantilever beam in Fig. 6 is designed against yielding due to bending stress while the cross-sectional area is desired to be kept as minimum. Five random variables are considered, including two design variables: $\mathbf{x} = [x_1, x_2]^T = [b, h]^T$ and three design parameters $p = [p_1, p_2, p_3]^T = [R, Q, L]^T$. b and h are the dimensions of the cross-section, L is the length of the beam, and b, h, and L are all normally distributed. Q is the external force with an extreme value distribution. The extreme distribution is a limiting distribution for the smallest or largest values in large samples drawn from a variety of distributions. It is used to represent the load which is applied many times and whose maximum value is of interest. R is the allowable stress of the beam with a Weibull distribution.

Insert Fig. 6 here.

Parameter distributions are described in Table 1. In robust design, the variables to be determined are the mean values (m_{x1} and m_{x2}) *of b* and *h*.

Table 1 Distributions of Random Variable

Name	Symbol	Mean Value	Standard Deviation	Distribution Type
R	p_1	200Mpa	20Mpa	Two-parameter Weibull
Q	p_2	20KN	2KN	Extreme Value Distribution
L	p_3	0.2m	1.0mm	Normal
b	x_1	μ_{x1}	0.05mm	Normal
H	x_2	μ_{x2}	0.05mm	Normal

The maximum tensile stress is calculated as

$$S_{\max} = \frac{QLh/2}{I} = \frac{QLh/2}{bh^3/12} = \frac{6QL}{bh^2} \tag{3.2}$$

The strength requirement can then be defined by the following constraint:

$$g(\mathbf{x},\mathbf{p}) = R - \frac{6QL}{bh^2} = p_1 - \frac{6p_2 p_3}{x_1 x_2^2} \geq 0 \tag{3.3}$$

A 99.95% probability of this constraint satisfaction is required. Constraint boundaries are obtained by using various feasibility formulation methods (see Fig. 7). The feasibility direction is also indicated.

Insert Fig. 7 here.

It is noted that the location of the constraint curve obtained based on the moment matching formulation is different from that obtained from the probability feasibility analysis. This is true due to the nonlinearality of the constraint function that follows a non-normal distribution. For this particular problem, the moment matching method provides an infeasible solution.

When using the worst case analysis by assuming $\Delta x = 3s_{\mathbf{x}}$ and $\Delta p = 3s_{\mathbf{P}}$, we find that the formulation generates conservative results, especially over the design region where m_{x2} is large and m_{x1} is small. As for the corner space formulation and the variation pattern formulation, because the variations of the design variables x_1 and x_2 are very small, the obtained constraint curves (the curve of variation pattern is not shown in Fig. 7) are very close to the deterministic one. This indicates that the feasibility robustness evaluated by these two methods is not reliable for this particular example.

In terms of the objective of keeping the cross-sectional area the minimum, the cross-sectional area is expressed as

$$s = x_1 x_2 \tag{3.4}$$

In robust design, the objective robustness is achieved by minimizing both the mean value and the variance of the cross-sectional area:

$$m_s = m_{x1} m_{x2} \tag{3.5}$$

$$s_s^2 = m_{x1}^2 s_{x2}^2 + m_{x2}^2 s_{x1}^2 + s_{x1}^2 s_{x2}^2 \tag{3.6}$$

For feasibility, we also expect that the ratio of *h/b* should be less than 2 and use the nominal values to express the constraint function (see Eqn. 3.9). The robust optimization model is stated as:

Find: mean value (m_{x1} and m_{x2}) of *b* and *h*

$$\text{min} \quad F(\mathbf{x}, \mathbf{p}) = w_1 m_S / m_s^* + w_2 s_S / s_s^* \tag{3.7}$$

$$\text{s.t.} \quad P[p_1 - \frac{6 p_2 p_3}{x_1 x_2^2} \geq 0] \geq P_0 = 99.95\% \tag{3.8}$$

$$2 - \frac{m_{x2}}{m_{x1}} \geq 0, \tag{3.9}$$

where m_s^* and s_s^* are the best achievable optimal solution of m_s and s_s, respectively. Here we use the weighting factor method to formulate the multiple objective function, though other advanced mathematical programming approaches can be used to better illustrate the tradeoff between the mean and variance attributes (Chen, et al. 1999). For the purpose of illustration, we use weighting factors w_1=w_2=0.5. The optimal solutions from using with different formulations of the constraint in Eqn. (3.8) are presented in Table 2. They are also marked as 1, 2, 3, etc. in Fig. 7.

To compare the accuracy and to confirm the design feasibility for each different approach, the optimization solution from the general Monte Carlo simulation with simulation size N=10^5 is considered as an accurate solution. Note that for the size chosen, the error bound of the probability evaluation under the confidence level 95% by Monte Carlo simulation is 1.39×10^{-4} (obtained from Eqn. 2.8a). The proposed *MPP*-based sampling method is also used for the probabilistic feasibility formulation. Also listed in Table 2 are the average numbers of function calls for one feasibility evaluation when using different approaches.

Even though the deterministic optimization generates the lowest values of mean and variance of the cross-sectional area, its feasibility is the worst with only 53.6% possibility of constraint satisfaction. The worst case formulation obtains the most conservative result with the probability greater than the specified limit. On the other hand, the moment matching method obtains a solution with its actual probability of constraint satisfaction equal to 0.9852, which is less than the specified limit. The corner space formulation gives a solution that is very close to that from the deterministic formulation with slightly higher probability of constraint satisfaction. The probabilistic feasibility formulation generates the optimum result with the exact required probability of constraint satisfaction when using the *MPP* based importance sampling method. Compared with the general Monte Carlo simulation, the *MPP* based importance sampling method has the same accuracy but needs much less computational effort.

Table 2 Solutions of the Beam Example

Formulation method		(m_{x1}, m_{x2}) (mm)	Probability of strength constraint satisfaction	Mean value of cross-sectional area m_S (mm^2)	Standard deviation of cross-sectional area s_s (mm)	Average number of function calls for one feasibility evaluation
Probabilistic feasibility formulation	Monte Carlo	(39.42, 68.15)	0.9995	2686.47	3.9365	100000
	MPP based importance sampling	(39.25,68.71)	0.9995	2596.87	3.9565	200
Moment matching formulation		(34.91,69.81)	0.9852	2437.07	3.9026	6
Worst case formulation		(37.88,75.77)	0.9999	2870.17	4.2356	6
Corner space formulation		(31.22, 62.43)	0.5433	1949.06	3.4901	32
Conventional deterministic optimization		(31.07, 62.14)	0.5360	1930.69	3.4737	1

3.3 Pratt & Whitney (PW) Engine Design

The PW engine design is used in this study to illustrate the applicability of different approaches for problems that are in a complex domain and taking a lot of computational resources. The problem statement of the PW engine design problem is provided in (Varadarajan, et al., 2000). A total of five continuous design variables are considered as the to-be-determined top-level design specifications. They are: the Fan Pressure Ratio (FPR), the Exhaust Jet Velocity Ratio (VJR), the Turbine Inlet Temperature or the Combustor Exit temperature (CET), the High Compressor Pressure Ratio (HPCPR), and the Low Compressor Pressure Ratio (LPCPR). There are three design parameters, i.e., High Turbine Compressor Efficiency (ehpc), High Turbine Efficiency (ehpt), and Low Turbine Efficiency (elpt). Variations are considered for both the design variables and the design parameters, which are all normally distributed (see Table 3).

Table 3 Distributions of design variables and design parameters

Design variables	Design parameter	Mean	Standard Deviation	Range
FPR		μ_{FPR}	0.1	$1.25 \le \mu_{FPR} \le 1.6$
VJR		μ_{VJR}	100	$0.6 \le \mu_{VJR} \le 0.9$
CET		μ_{CET}	1.5	$2400 \le \mu_{CET} \le 4000$
HPCPR		μ_{HPCPR}	0.5	$10.2 \le \mu_{HPCPR} \le 25$
LPCPR		μ_{LPCPR}	0.1	$1.15 \le \mu_{LPCPR} \le 4.9$
	Ehpc	0.891	$10^{-3}/3$	—
	Ehpt	0.933	$10^{-3}/3$	—
	Elpt	0.9	$10^{-3}/3$	—

The task is to find the mean values of design variables, FPR, VJR, CET, HPCPR, and LPCPR, to maximize the mean aircraft range (μ_{RANGE}) and to minimize the corresponding standard deviation (σ_{RANGE}) simultaneously. The objective is expressed by the weighting factor method as

$$\min \quad w_1\left(-\frac{\mu_{\text{RANGE}}}{3000}\right) + w_2 \frac{s_{\text{RANGE}}}{10} , \qquad (3.10)$$

where w_1 and w_2 are weighting factors ($w_1 = 0.5, w_2 = 0.5$) and the values 3000 and 10 are used to normalize the mean and the standard deviation of RANGE, respectively.

All the system outputs RANGE, OPR (Overall Pressure Ratio), FANDIA (Fan Diameter), and HPTPR (High Turbine Pressure Ratio) are nonlinear functions of design variables and noise factors (Varadarajan, et al., 2000). The probabilistic constraint satisfaction is modeled as:

$$P\ (\text{OPR} \le 30) \ge 0.98 \qquad (3.11)$$

$$P\ (\text{FANDIA} \le 80) \ge 0.985 \qquad (3.12)$$

$$P\ (\text{HPTPR} \le 3.5) \ge 0.99 \qquad (3.13)$$

The robust design results by using different feasibility modeling methods are provided in Table 4. To compare the effectiveness of different methods, the result of the conventional deterministic model (without the consideration of any uncertainty) and the result from using the general Monte Carlo simulation are also listed. A large simulation size $N=10^5$ is utilized for this example problem in the general Monte Carlo simulation. If the confidence level is taken as 95%, at the optimum point, the error bound of the probability of constraint satisfaction calculated by Eqn. (2.8a) becomes 8.6773×10^{-4} for OPR constraint, 7.0208×10^{-4} for FANDIA constraint, and 3.3897×10^{-4} for HPTPR constraint. Therefore the result from the general Monte Carlo simulation is accurate enough to be considered as the true result for confirmation purpose.

Table 4 Solutions of the Engine Design Problem

Method		Optimum solution			Probability of constraint satisfaction			Number of unsatisfied constraints	Average number of function calls for one feasibility evaluation
		Means of design variable	μ_{RANGE}	σ_{RANGE}	OPR	FANDIA	HPTPR		
Probabilistic feasibility formulation	Monte Carlo	0.8248,2811.6197, 10.3282,1.7741, 1.5933	2224.5438	55.4104	0.980 =0.98	0.987 >0.985	0.997 >0.99	0	100,000
	MPP based importance sampling	0.8249,2811.4418, 10.3330,1.7761, 1.5933	2224.6832	55.2079	0.9803 >0.98	0.9887 >0.985	0.9960 >0.99	0	240
Moment matching formulation		0.8269,2818.9474, 10.2,1.8487, 1.5998	2229.4976	48.0750	0.9725 <0.98 violated	0.9910 >0.985	0.9980 >0.99	1	9
Worst case formulation		0.8478,2881.9063, 10.2045,1.150, 1.60	2220.5225	104.1189	0.9985 >0.98	0.9895 >0.985	1.0>0.99	0	9
Corner space formulation		0.6,2400.0, 10.2045,1.15, 1.5998	2065.9401	155.2793	0.9985 >0.98	1.0 >0.985	0.9950 >0.99	0	256
Conventional deterministic formulation		0.7669,4000.0, 20.8630,1.4380, 1.60	2681.1454	61.7006	0.50 <0.98 violated	0.9905 >0.985	0.8735 <0.99 violated	2	1

The general Monte Carlo method (Eqn. 2.6) is also used to evaluate the actual probabilities of constraint satisfaction at those optimum points generated from different feasibility formulations. In Table 4, under each entry for OPR, FANDIA and HPTPR, the top row is the actual probability value and the bottom row is the desired probability value. Under the category of probabilistic feasibility formulation, the results from using the MPP based method are listed to illustrate the improved efficiency of our proposed approach.

We note that the robust design solution including the mean and variance of RANGE obtained by the MPP based method is very close to that from the general Monte Carlo simulation. The constraint satisfaction also matches well with the real situation in which OPR is an active constraint (Eqn. 3.11), and the remaining two constraints are not active. The optimum solution from the moment matching formulation is close to that from the general Monte Carlo simulation; however, it is infeasible because constraint OPR is slightly violated. The worst case formulation generates a feasible but conservative solution. The obtained standard deviation of RANGE (104.1189) is much bigger than the one from the probabilistic feasibility formulation (55.4104 or 55.2079) and the corresponding mean of RANGE is less. The accuracy of the worst case formulation is even worse compared to the conventional deterministic formulation. The robust design solution obtained from the corner space formulation is far away from the rest of the solutions. For this particular problem, it suffers the same shortcoming as the worst case formulation. As predicted, the conventional deterministic formulation generates the lowest probability for constraint satisfaction. OPR constraint is satisfied with only 0.5 probability. The probability of HPTPR constraint satisfaction is also less than the required value.

To compare the efficiency of different approaches, the average number of function calls for one feasibility evaluation is provided in the last column of Table 4. The *MPP* based importance sampling method needs fewer number of simulations than the general Monte Carlo simulation while they have the similar accuracy. We further confirm that if the same amount of simulations (240) is used for the general Monte Carlo simulation, the results of constraint satisfaction become 0.9460 (for OPR constraint), 0.9915 (for FANDIA constraint), and 0.9915 (for HPTPR constraint). This is not even a feasible solution. For this example problem, the number of function calls for each feasibility evaluation is at the low end for the moment matching method and the worst case formulation. Because the number of design variables is more than that of the beam design problem in Section 3.2, the corner space formulation requires more function evaluations compared to the *MPP* based importance sample method. This indicates the gain of the *MPP* based importance sample method for large scale problems. However, we should also note that when there are too many random variables (for example, over 50), the computation burden of the *MPP* based importance sample method will increase due to the optimization procedure used for searching the *MPP*.

3.4 A Summary of Comparisons

Based on the three example problems presented in Sections 3.1, 3.2, and 3.3, the features of various existing methods for modeling feasibility robustness are summarized and compared in Table 5. We have included various attributes in this comparison, such as whether the constraint function requires statistical evaluation, whether the description of uncertainty distribution has to be given, how the performance distributions are described, and whether the calculation of partial differential of the function is needed, etc. The number of function evaluations required and the capacity and accuracy of each method are also summarized.

Table 5 Comparisons of Feasibility Modeling Techniques

	Probabilistic Feasibility Formulation	Moment Matching Formulation	Worst Case Analysis	Corner Space Evaluation	Variation Pattern
Require statistical evaluation	Yes	Yes	No	No	No
Description of uncertainty distribution	Necessary	Not necessary	Not necessary	Not necessary	Necessary
Description of performance distribution	Yes	Only mean value and standard deviation	Extreme values	Extreme values	Extreme values
Deal with correlation	Yes	No	No	No	Yes
Calculation of derivatives	May or may not	Yes	Yes	No	No
Number of constraint function evaluations (*N*)	• N=1 when the analytical probability can be derived; • N = simulation number for Monte Carlo simulation; • N = function call of *MPP* search + number of samples for *MPP* based importance sampling.	Evaluation includes mean and variance (function differentiation). $N=m+n+1$.	Evaluation includes mean and variance (function differentiation). $N=m+n+1$.	Evaluation includes calculating function values at the "corners". $N = 2^{m+n}$.	Evaluation involves searching the tangent point of $MVP(1\text{-}a)$ with the original constraint boundary. N depends on the shape of the variation pattern.
Capability and accuracy	• Gives accurate probability estimation; • Solve complicated problems; • Difficult to get analytical solution; • Needs great computational effort especially when simulations are involved.	• Gives exact probability estimation for normally distributed functions; • Provides approximations for other problems; • The accuracy of result is sensitive to the mathematical structure of the constraint.	• Simple to use; • Low estimation accuracy; • In most cases, gives over-conservative results.	• Simple to use; • Calculation amount increases with variable dimension increasing. • Doesn't provide the probability (level) of constraint satisfaction.	• More accurate than the corner space method; • Complicated to use in the process of optimization; • Doesn't provide the probability (level) of constraint satisfaction.

In summary, if neglecting the computational effort, the probabilistic feasibility formulation using either analytical derivations or Monte Carlo simulations is the ideal method to describe the feasibility robustness that can ensure the solution achieves an accurate level of constraint satisfaction. However, in general, it is very time-consuming to evaluate the probability of constraint satisfaction using the general Monte Carlo simulations. If using the proposed *MPP* based importance sampling method, the sampling number will be reduced significantly. If the calculation cost is more concerned by designers, the alternative formulations, such as the moment matching formulation, should be considered. The moment matching formulation provides an accurate estimation of the probability when the performance is normally distributed. On the other hand, we should pay attention to the fact that when using different mathematical structures for the same constraint function, different results may be obtained by the moment matching method due to the differences which exist in the first-order Taylor's expansion (Chen and Weng, 1998).

The methods in the category of "not requiring probability and statistical analyses" better suit the problems in which the distributions of variables and parameters are not available. The worst case formulation is a good selection under this situation. Though the worst case analysis is widely considered as a conservative approach for modeling feasibility robustness, we should use it with caution since the violation of the constraint is still possible over certain design regions. To avoid statistical analysis or the evaluation of partial differential of constraint functions, the methods of corner space formulation and variation pattern can be adopted. The accuracy of these methods depends on whether the constraint function is monotonic with respect to all design variables in the tolerance space and whether the tolerance of design variables are the only source of variation. One limitation is that these two methods do not provide the information on the probability (level) of constraint satisfaction.

If computational efficiency is the main criterion for selecting an appropriate approach, the number of constraint function evaluation listed in Table 5 provides a good guideline. While the total number of constraint function evaluations can be predicted in a generic way with respect to the total number of random variables for the moment matching formulation, the corner space evaluation, and the worst case analysis (See Table 5), we are not able to express this relationship for the *MPP* based importance sampling method and the variation pattern method because of the optimization-based search procedure involved. Generally, the moment matching formulation and the worst case analysis require the least amount of function evaluations, while the general Monte Carlo simulation requires the most. When the number of random variables is not so large, the corner space formulation and the variation pattern formulation may be more efficient than the *MPP*-based importance sampling. But for large number of random variables, the situation will be opposite. There are some special cases. For example, when the probability can be derived analytically, then the probabilistic formulation will be the most efficient. While the errors of the majority of the methods cannot be predicted ahead of time, the errors of the general Monte Carlo simulation and the *MPP*-based importance sampling can be evaluated for specified confidence levels (see discussions in Section 2.4). For the problem with very large number of random variables (for example, hundreds and thousands), the advantage of *MPP* based importance sampling method over the general Monte Carlo simulation will diminish due to the optimization procedure used for searching for the *MPP*.

4. CLOSURE

In robust design, it is important not only to achieve the robust objective performance but also to maintain the robustness of design feasibility. In this paper, we discussed how to define the robustness of design feasibility under the effect of variations. By providing analytical interpretations and using illustrative examples, the features of various existing methods for modeling feasibility robustness are compared from different aspects. We illustrate that, although some of these approaches are easy to use, they may lead to either conservative or infeasible design solutions in robust design applications. The summary of comparisons is provided in Section 3.4 and will not repeat here. We expect that they could serve as guidelines for choosing the right technique under different circumstances.

It is our belief that the probabilistic feasibility formulation is the ideal method to describe the feasibility robustness and to ensure the solution achieve an accurate level of constraint satisfaction. To improve the efficiency of using this formulation, we propose to use the most probable point (*MPP*) based importance sampling method, a technique rooted in reliability analysis, for evaluating the feasibility robustness. Though our discussions have been centered on robust design, the principles presented are generally applicable for any probabilistic optimization problems. The practical significance of this work also lies in the development of efficient feasibility evaluation methods that can support quality engineering practice, such as the Six Sigma approach that is being widely used in American industry.

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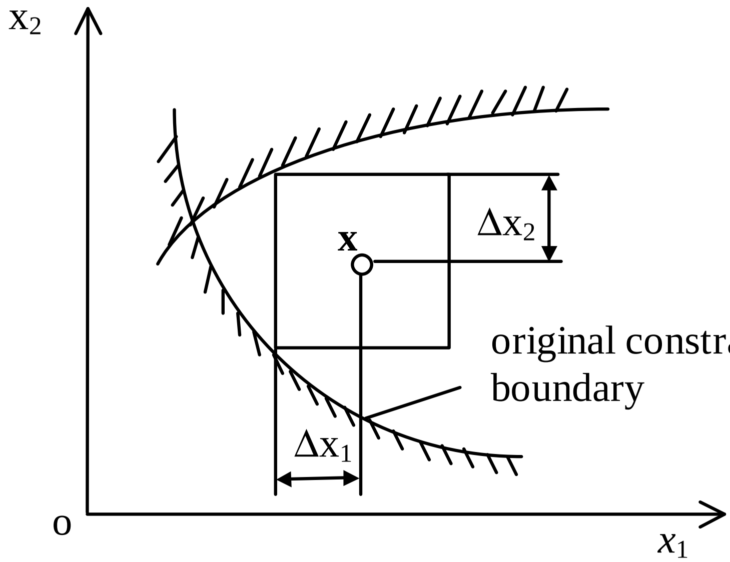


Figure 1. Feasibility under the Corner Space Evaluation Method

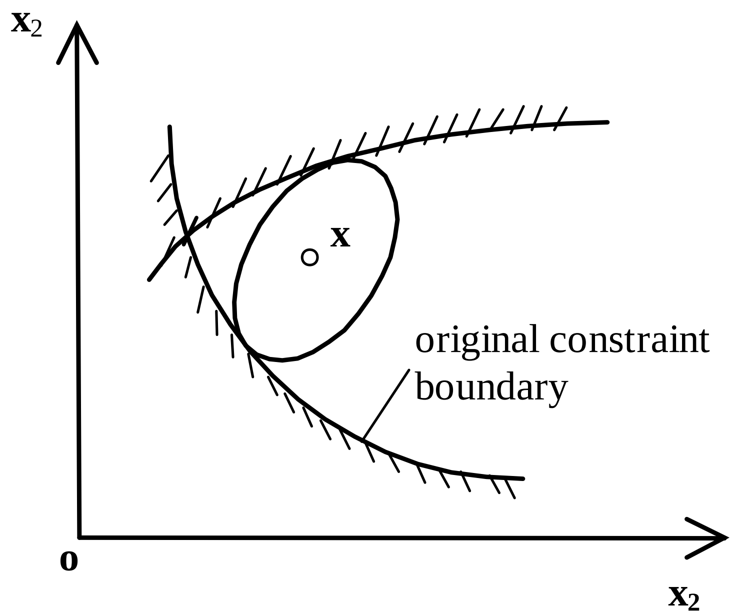


Figure 2. Variation Pattern Analysis Method

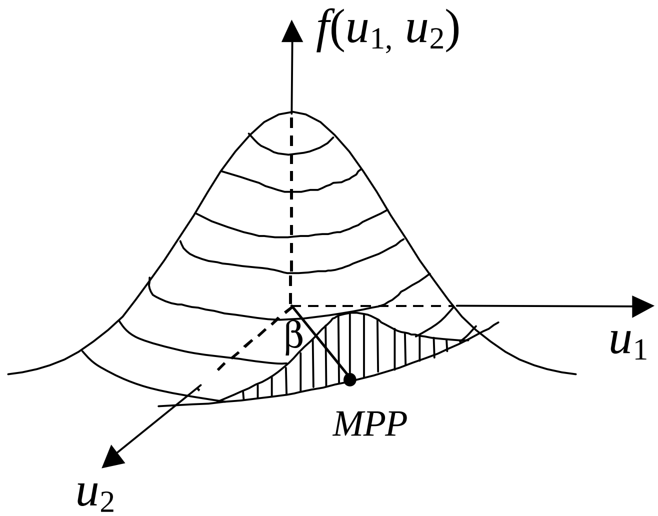


Figure 3 The *MPP* concept

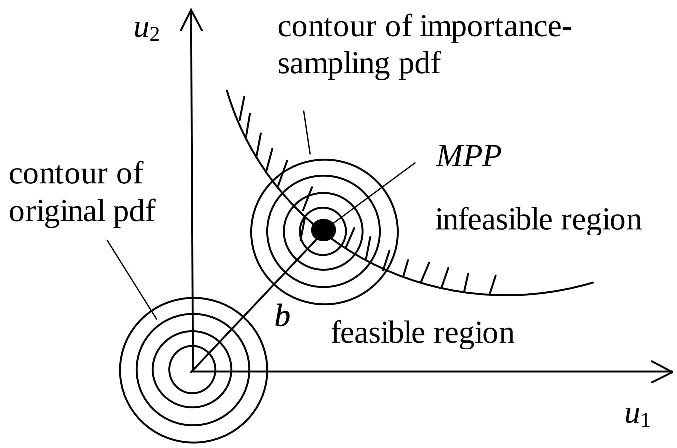


Figure 4. Importance Sampling

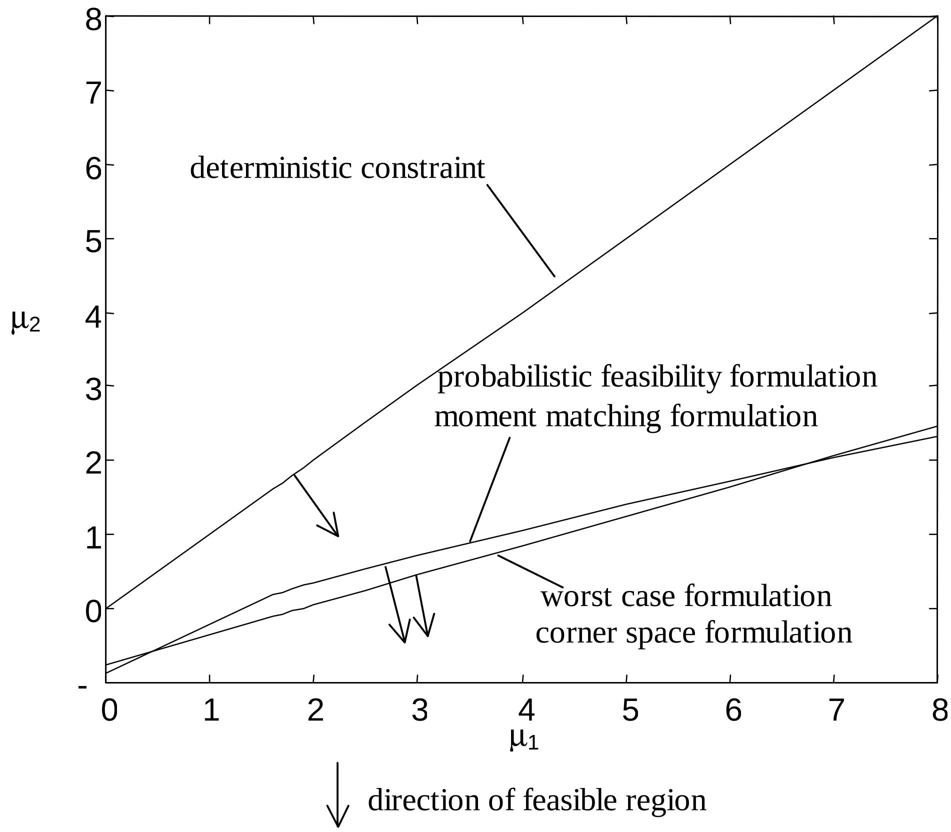


Figure 5. Comparisons of Feasibility Curve

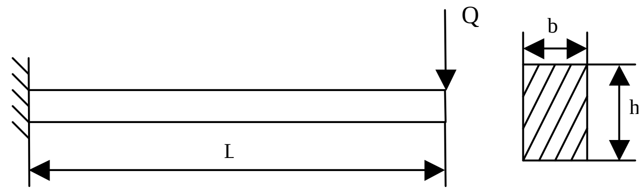
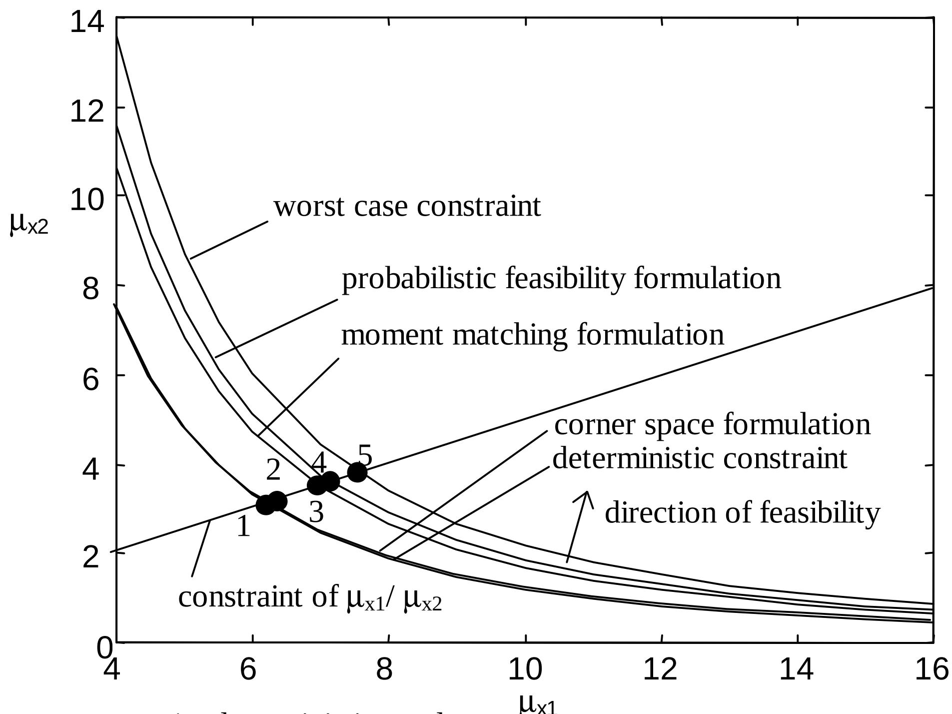


Figure 6. Cantilever Beam



1 - deterministic result
2 - result from corner space
3 - result from moment matching formulation
4 - result from probabilistic formulation Method)
5 - result from worst case formulation

Figure 7. Comparisons of Feasibility Analyses