# A MOST PROBABLE POINT BASED METHOD FOR UNCERTAINTY ANALYSIS


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# ABSTRACT

Uncertainty is inevitable at every stage of life cycle development of a product. To make use of probabilistic information and to make reliable decisions by incorporating decision maker's risk attitude, methods for propagating the effect of uncertainty are therefore needed. When designing complex systems, the efficiency of methods for uncertainty propagation becomes critical. In this paper, a most probable point (*MPP*) based uncertainty analysis (*MPPUA*) method is proposed. The concept of the *MPP* is utilized to generate the cumulative distribution function (*CDF*) of a system output by evaluating probability estimates at a serial of limit states across a range of output performance. To improve the efficiency of locating the *MPP*, a novel *MPP* search algorithm is presented that employs a new search algorithm and a search strategy. A mathematical example and the Pratt & Whitney (PW) engine design are used to verify the effectiveness of the proposed method. The proposed *MPPUA* method will serve as a useful tool for propagating the effect of uncertainty when making decisions under uncertainty.



Key words: Uncertainty analysis, most probable point, probabilistic distribution, decision making under uncertainty.

# NOMENCLATURE

| | |
|---|---|
| *CDF* | cumulative distribution function |
| *MPP* | Most Probable Point |
| *MSC* | Monte Carlo Simulation |
| *PDF* | probability density function |
| *F* | a *CDF* |
| *f* | a *PDF* |
| *g* | a model function |
| *N* | number of simulation |
| *n* | number of design variables |
| *P* | probability |
| **u** | standard normal random vector |
| **x** | model inputs |
| *y* | model output |
| *z* | limit-state function |
| *F* | *CDF* of standard normal distribution |
| *b* | safety index |
| *m* | mean value |
| *r* | main curvature |
| $s^2$ | variance |
| $\|\cdot\|$ | norm of a vector |

## 1. INTRODUCTION

The notion that engineering design is a decision-making process has been paid more and more attention in recent years [1-5]. When uncertainty is considered in engineering design, probabilistic design models are adopted instead of deterministic models. Unfortunately, probabilistic models require order(s) of magnitude increase in the analysis in comparison to the deterministic models [6]. The problem is worse when a design model involves complex computations, such as calculations of stresses, temperatures, heat transfer rates, and fluid flows through numerical algorithms or simulation tools that could involve finite element analysis, computational fluid dynamics, etc. One of the challenges to use probabilistic design models is to capture the effect of uncertainty on the system output in an efficient manner. The problem can be stated as: *given the probability distributions of the random variables* in a system (e.g., those of the *design variables and parameters*), *what should be the probability distributions of the system output*? This process is often referred to as uncertainty propagation or uncertainty analysis. Our aim in this paper is to develop efficient methods for uncertainty analysis that are practically affordable in engineering design applications.

Once the effect of uncertainty is propagated, the information on system output distribution will be used further for making reliable decisions through optimization. The decision-based design (DBD) [3] is such a framework that provides the capability to make use of probabilistic information and to incorporate decision maker’s risk attitude under uncertainty through utility function optimization.

Researches on the sources of uncertainty and how to model them can be found in the literature (the recent references including [7-10]. The commonly used method for uncertainty analysis is the sensitivity-based approximation approach that includes the worst case analysis and the moment matching method [11-14]. With the worst case analysis, all the fluctuations are assumed to occur simultaneously in the worst possible combinations and based on this assumption, the worst value of the system output (extreme condition) can be found by the first order Taylor expansion or optimization. With the moment matching method, the first order moment (mean value) and the second order moment (standard deviation) of a system output are obtained. However, the moment matching method is generally not sufficiently accurate, especially when the parameter uncertainty is large. Besides, low order moments do not provide the shape of the resulting probability distribution.

Compared with the sensitivity-based approximation approach, Monte Carlo Simulation (*MCS*) is a more comprehensive method that can generate the cumulative distribution function (*CDF*) and the probability density function (*PDF*) of a system output based on data sampling. The shortcoming of *MCS* is that great computational effort is required for any general cases. Some modified *MCS* methods have been proposed to improve the computational efficiency. Among them are the importance sampling method [15,16], the Latin hypercube sampling method [17], the shooting Monte Carlo approach [18], and the directional simulation [19]. However, even with modifications, MCS is generally not affordable in the design of complex engineering systems.

As the direct use of data sampling methods is very expensive, an alternative approach that is being widely considered is the use of *response surface models* to replace the numerical models [6, 20-23], either over the space of all variables or that of the random variables only [24,25]. Data sampling is then performed based on the response surface model instead of the original numerical model. The drawback of using this approach is the cost associated with generating an accurate response surface model for the purpose of uncertainty propagation. Besides, some of the response surface methods tend to "smooth" the behaviors. This results in large errors when the smooth function is used to evaluate local sensitivities in probability analysis.

Reliability analysis based approaches have emerged as efficient uncertainty analysis methods with a lot of promises 26,27]. They generally provide much better accuracy compared to sensitivity-based approximations and response surface modeling approaches, and much better efficiency compared to sampling-based methods. Reliability analysis methods are characterized by the use of analytical techniques to find a particular point in design space that can be related (at least approximately) to the probability of system failure, defined by a limit state [28] This point is often referred to as the most probable point (*MPP*) or the design point. The typical methods in the filed of reliability analysis for probability estimation include FORM (first order second moment method) and SORM (second order reliability method) [28]. Details of reliability analysis method are provided in Section 2.2.

In this paper, we propose an efficient method for generating the probabilistic distributions of a system output based on the concept of the most probable point (*MPP*). We call this method the most probable point based uncertainty analysis (*MPPUA*) method. To overcome the limitations of the existing *MPP* search methods, a novel *MPP* search approach is presented that employs a set of searching strategies, including using a new inverse algorithm, tracing the *MPP* locus, and predicting the initial search point.

This paper is organized as follows. In Section 2, the frame of references of our research is presented. The concepts of Monte Carlo Simulations and the most probable point (*MPP*) are introduced respectively. The proposed *MPPUA* method is presented in Section 3 along with the discussions of an *MPP* search algorithm and other search strategies. In Section 4, two examples are used to demonstrate the effectiveness of the proposed method. The accuracy and efficiency of the *MPPUA* method are examined. Section 5 is the closure of this paper.

## 2. FRAME OF REFERENCES

A generic uncertainty analysis problem is considered in this work. We let model inputs $\mathbf{x} = \{x_1, x_2 . \cdots x_n\}$ be mutually independent random variables with the cumulative distribution function (*CDF*) $F_i(x_i)$ and the probability density function (*PDF*) $f_i(x_i)$ ($i = 1, n$). The system model $g(\mathbf{x})$ maps model inputs $\mathbf{x} = \{x_1, x_2 . \cdots x_n\}$ into an output $y$ as:

$$y = g(\mathbf{x}). \tag{1}$$

Analytical methods can be used to generate the *PDF* of the output *y*. However they are difficult or even impossible to use for complex system behaviors. Simulation methods are widely used whenever the analytical methods are not practical.

### 2.1 Monte Carlo Simulation (*MCS*)

*MCS* is used for the purpose of confirmation in this paper. Results from the proposed method will be compared with those from *MCS* to show the effectiveness of the proposed method. The *CDF* of $y$ at $y_i$ ($i = 1, K$) is defined by the following probability:

$$F_y(y_i) = P\{y < y_i\} = P\{g(\mathbf{x}) < y_i\}. \tag{2}$$

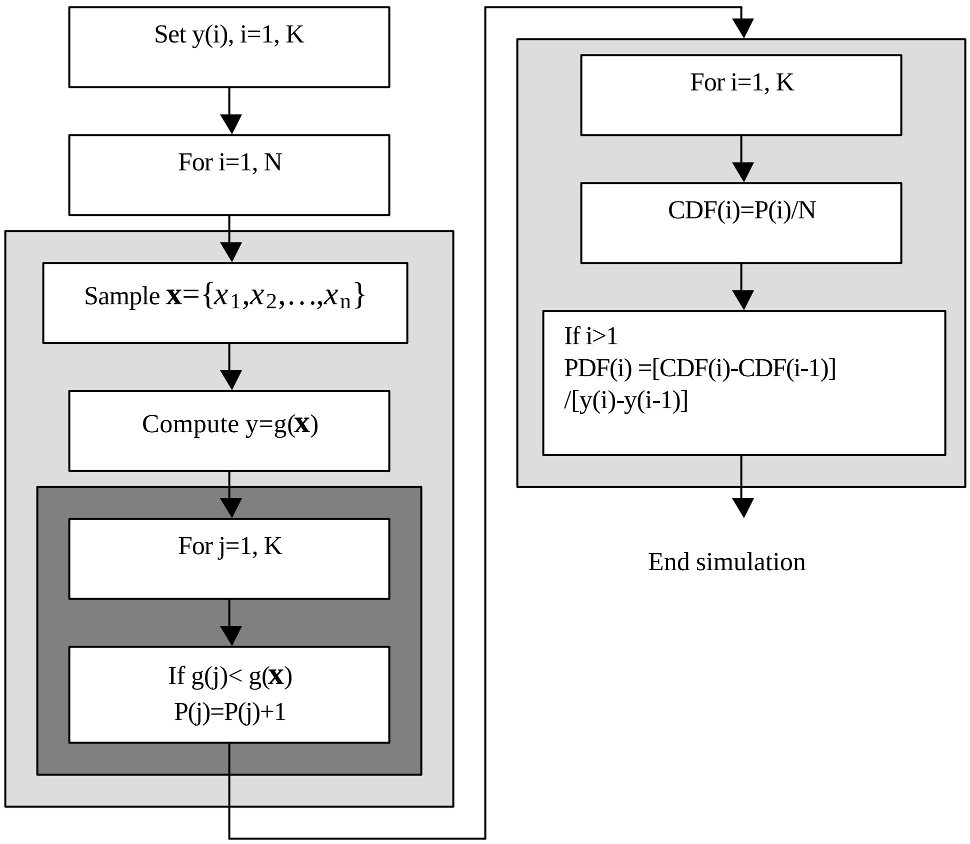



**Figure 1 Monte Carlo Simulation**

The procedure of *MCS* is shown in Fig. 1. With *MSC*, the samples of input variables $\mathbf{x}$ are generated based on their cumulative distribution functions $F_i(x_i)$ $(i = 1, n)$ or their probability density functions $f_i(x_i)$. A system output $y$ is then evaluated at each $\mathbf{x}$ sample. The *CDF* of $y$ at $y_i$ $(i = 1, K)$ is estimated by the frequency of $y$ samples less than $y_i$ shown as follows:

$$F_y(y_i) = \frac{\text{Number of } y \text{ samples less than } y_i}{N}, \tag{3}$$

where $N$ is the total number of simulations.

The *PDF* of $y$ at $y_i$ can be evaluated by the differential of *CDF*:

$$f_y(y_i) = \frac{F_y(y_i) - F_y(y_{i-1})}{y_i - y_{i-1}} . \quad (4)$$

*MCS* is flexible for any types of input distributions and any forms of model functions. Neglecting the algorithmic error caused by simulations, if a sufficient number of simulations $N$ are used, *MCS* often results in solutions with a high accuracy. Compared with other numerical methods, *MCS* has a remarkable feature that its accuracy does not depend on the dimension of the random model input variables (Niederreiter, 1992). However, in general, it is time consuming to obtain accurate enough results via *MCS*.

### 2.2 The Most Probable Point (*MPP*) Method

The *MPP* method was originally developed in the field of reliability analysis [29]. It requires that limit-state functions be defined and then the probability of the limit-state functions bigger or less than zero can be evaluated approximately. The limit-state function is defined as:

$$z(\mathbf{x}) = y - c = g(\mathbf{x}) - c , \quad (5)$$

where $c$ is a constant.

The *MPP* is formally defined in a coordinate system of an independent and standardized normal vector $\mathbf{u} = \{u_1, u_2, \cdots u_n\}$. The input variables $\mathbf{x} = \{x_1, x_2 .\cdots x_n\}$ (in the original design space) are transformed into the standard normal space $\mathbf{u} = \{u_1, u_2, \cdots u_n\}$. The most commonly used transformation is given by Rosenblatt [30] as

$$u_i = \Phi^{-1}[F_i(x_i)] \qquad (i = 1, n), \tag{6}$$

where $\Phi^{-1}$ is the inverse of a normal distribution function. The transformation maintains the *CDF*s being identical both in **x** space and **u** space (see Figure 2).

The limit-state function is now rewritten as

$$z(\mathbf{u}) = g(\mathbf{u}) - c. \tag{7}$$

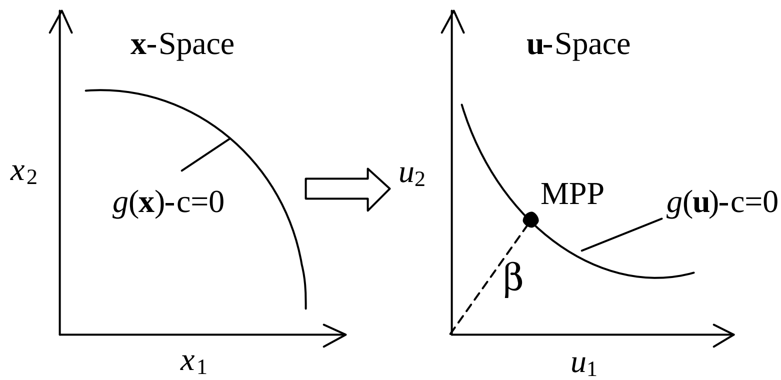



**Figure 2. The Transformation of input variables**

Hasofer and Lind [29] defined $b$ as the shortest distance from the origin to a point on the limit-state surface in **u** space (see Fig. 2). Mathematically, it is a minimization problem with an equality constraint:

$$b = \min_{\mathbf{U}} \|\mathbf{u}\| \tag{8}$$

$$\text{subject to } g(\mathbf{u}) - c = 0. \tag{9}$$

The solution $\mathbf{u}_{MPP}$ of this minimization problem is called the most probable point (*MPP*). From Fig. 3, we see that the joint probability density function on the limit-state surface has its highest value at the *MPP* so that *the MPP in the standard normal space has the highest probability of producing the value of limit-state function* $z(\mathbf{u})$ [31]. In other words, it is the point that contributes the most to the integral for probability estimation $b$ is also referred to as the safety index in reliability analysis and the MPP becomes the critical design point.

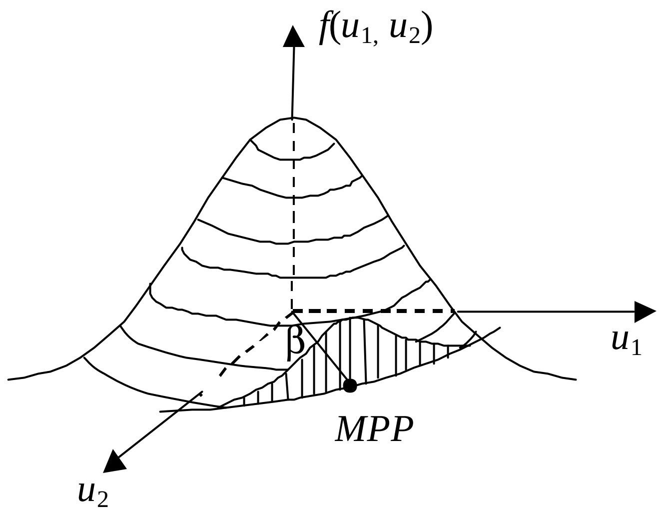



**Figure 3. The *MPP* concept**

If the limit-state function $z(\mathbf{x})$ or the model function $g(\mathbf{x})$ is linear in terms of the normally distributed random variables $\mathbf{x}$, the accurate probability of limit-state less than zero is given by the equation:

$$P\{z(\mathbf{u})<0\}=P\{g(\mathbf{x})<c\}=\begin{cases}\Phi(b) & \text{if } P\geq 0.5\\ 1-\Phi(b) & \text{if } P<0.5\end{cases} \quad (10)$$

If $g(\mathbf{x})$ is nonlinear or random variables $x_i$ $(i=1,n)$ are not normally distributed, a good approximation can still be obtained by the above equation, provided that the magnitude of the principal curvatures of the limit-state surface in the $\mathbf{u}$ space at the *MPP* is not too large [27]. If the limit-state function is highly nonlinear, an alternative second-order approximation at the *MPP* can be used, which takes into account the curvature of the limit-state surface around the *MPP* [32, 33].

## 3. A MPP BASED METHOD FOR UNCERTAINTY ANALYSIS

In this work, a most probable point (*MPP*) based uncertainty analysis (*MPPUA*) method is developed to estimate the probabilistic distribution of a system. The concept of the *MPP* is utilized to generate the cumulative distribution function (*CDF*) of a system output by evaluating probability estimates at a serial of limit states across a range of output performance. A novel MPP search algorithm is also developed to support the procedure of *CDF* generation.

### 3.2 Generating probabilistic distribution of a system output

The basic idea of the *MPPUA* method is to utilizes reliability assessments ($P\{y = g(u) = c\}$) at multiple limit states $c_i$ to obtain the probability estimates $P_i$ across a range of performance $y$. The definition of the *CDF* of a system output is

$$F_y(c) = P\{y < c\} = P\{g(\mathbf{u}) < c\}, \qquad (11)$$

which is equivalent to

$$F_y(c) = P\{g(\mathbf{u}) - c \le 0\} = P\{z(\mathbf{u}) < 0\}. \qquad (12)$$

The analogy is if we could identify the *MPP* for a system output $y$ (or g(**u**)) at a limit state $c$, this will provide us an estimate of the *CDF* of $y$ when $y = c$. For each MPP, there is a corresponding value of β (the shortest distance). Based on Eqn. 10, the *CDF* of $y$ at $y = c$ becomes

$$F_y(c) = P\{y < c\} = P\{z(\mathbf{u}) < 0\} = \begin{cases} \Phi(b) & \text{if } P \ge 0.5 \\ 1 - \Phi(b) & \text{if } P < 0.5 \end{cases}. \qquad (13)$$

If we could evaluate the *CDF* values using this way for a set of $c$ values chosen in a range of system output $y$, then we will be able to obtain a (discretized) complete *CDF* across a range of $y$. Since the reasonable range of a system output is generally not known in advance, difficulty will arise when choosing the set of *c* values. To overcome this difficulty and to ensure that the generated *CDF* roughly occupies the complete interval of [0, 1], we utilize an inverse approach to implement the proposed concept. Specifically, we will identify the limit-state values $c_i$ in the domain of a system output y, for a given set of probability levels $\{F_{y1}, F_{y2}, \cdots, F_{yK}\}$ using the *MPP* concept. To facilitate the *MPP* search procedure introduced in the later sections, instead of discretizing the probability levels, we choose to discretize the corresponding $b$ values into equal segments. A serial of $b$ *is* chosen as*:*

$$b^i = \frac{b_{\max}}{M} \ (i = 1, M), \tag{14}$$

where M is the number of points picked. $b$ is selected in the range of [0, 3] or[0, 4], which corresponds to *CDF* values in the range of $[0.0013, 0.9987]$ or $[3.1671 \times 10^{-5}, 0.99997]$, respectively. Geometrically as shown in Figure 4, in the standard normal space **u**, we will consider a family of concentric spheres with radius $b^i$ and we call these spheres $b$-spheres.

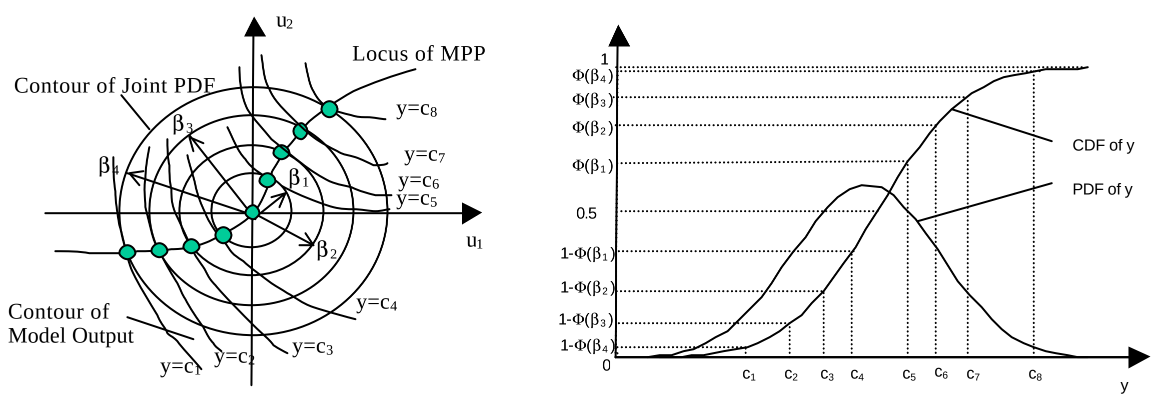



**Figure 4. Concept of The *MPPUA* Method**

Once the $b^i$ $(i = 1, M)$ are specified, the corresponding *CDF* of y are calculated by Eqn. 13. Since every $b^i$ yields two values of $F_{yi}$ except for $F_{yi}$ = 0.5, i.e., $b = 0$ (Eqn. 13), the total number of $F_{yi}$ and that of $b^i$ have the following relationship

$$K = 2M + 1, \tag{15}$$

where K is the total number of $F_{yi}$.

Note that the *MPP*s are identified in the standard normal space **u** and they should be transformed into the initial **x** space. Once the **x** values of the *MPP* are identified, $y_i$ can then be evaluated using the system function g(**x**). As shown in Figure 4, for each *b* sphere, two *MPP*s will be identified, corresponding to two output levels.

Based on the proceeding discussions, we summarize the procedure of generating the *CDF* of a system output *y* as follows:

(1) Specify a set of $b^i$ index $(i = 1, M)$;

(2) Calculate the two corresponding *CDF* values using Eqn. 13.

(3) For a given $b^i$, find two *MPPs* $\mathbf{u}^i_{MPP}$ in **u** space (details see Section 3.2);

(4) Map $\mathbf{u}^i_{MPP}$ into **x** space to obtain $\mathbf{x}^i_{MPP}$ using

$$x^i_{MPP,\,j} = F_i^{-1}[\Phi(u^i_{MPP,\,j})] \qquad (i = 1, K;\ j = 1, n). \tag{16}$$

(5) Evaluate the value of a system output at a *MPP by*

$$y_i = g(\mathbf{x}^i_{MPP}) \quad . \tag{17}$$

Once the discretized *CDF* is obtained, we can then obtain the *PDF*: $(f_y(y_i), y_i)$ using Eqn. 4.

### 3.2 Searching the *MPP*

From the above procedure, we see that the key to generate the correct probabilistic distribution function is to find the right *MPPs* for different $b^i$. Various techniques have been proposed in the literature to search the *MPP*, such as the hypersphere method [33], the directional cosines method [34], the advanced mean value (*AMV*) [31], the sampling-based *MPP* search [26], and the optimization method. Unfortunately, there has been no single search algorithm that could succeed to find the right *MPP* for all situations. As stated by Tichy [33], the common problems are: 1) The *MPP* obtained appears logical, but it is incorrect, as the iteration leads to a local extreme of the transformed limit-state functions; 2) Solution does not converge. For example, the search process may oscillate between two points. The newly developed sampling based *MPP* searching algorithm [26] seems to be highly robust, but it needs many iterations and much sampling to obtain the correct *MPP*.

In this paper, we develop a new algorithm that uses the derivative information of the limit-state function to direct the search and traces the *MPP* locus step by step from the center of a distribution to its tails. The algorithm is expected to be efficient and robust as it suits the nature of our proposed procedure in which probability levels are identified incrementally. Two issues are involved in the proposed algorithm. One is the *MPP* search algorithm for a specific $b^i$ and the other is the determination of the initial point for each search. We will discuss them in detail next.

#### 3.2.1 Searching the *MPP* at $b^i$

The idea behind our *MPP* search algorithm is to locate the tangent point of the hypersphere (called $b$-sphere with radius $b^i$) and the limit-state surface in the $\mathbf{u}$ space.

At this tangent point, the vector $\mathbf{d}^i$ connecting the *MPP* $\mathbf{u}^i_{MPP}$ and the origin **o** should overlap with the gradient $\nabla g(\mathbf{u}^i_{MPP})$ of the function $g(\mathbf{u})$ (see Fig. 5). This condition is valid for both *MPP*s on each β sphere, no matter what is the curvature (convex or concave) of the limit state.

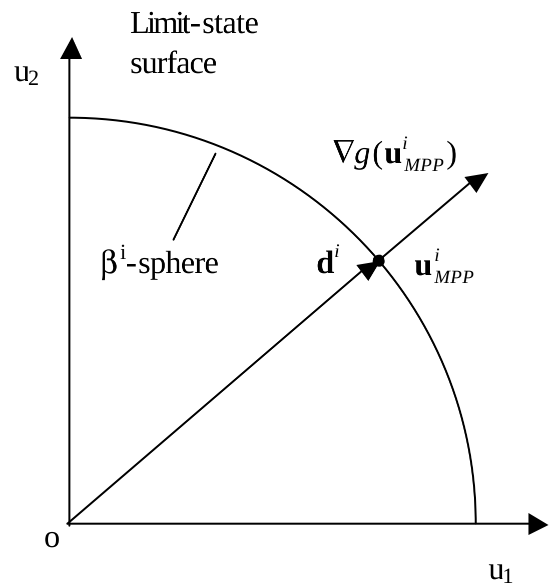



**Figure 5. Overlapping Condition at *MPP***

If we let the current point be $\mathbf{u}^i_k$ on the $b^i$-sphere (k stands for the k*th* iteration in searching the *MPP* that corresponds to $b^i$), our proposed search process is stated as follows:

(1) Set the vector connecting $\mathbf{u}^i_k$ and the origin **o** as:

$$\mathbf{d}^i_k = \mathbf{u}^i_k . \tag{18}$$

(2) Calculate the gradient $\nabla g(\mathbf{u}^i_k)$ at $\mathbf{u}^i_k$;

(3) If $a^i_k = \cos^{-1} \dfrac{\mathbf{d}^i_k \bullet \nabla g(\mathbf{u}^i_k)}{\left\|\mathbf{d}^i_k\right\| \cdot \left\|\nabla g(\mathbf{u}^i_k)\right\|} \le e_a$, $\mathbf{u}^i_{MPP} = \mathbf{u}^i_k$ and stop;

else, goto (4);

$e_a$ is a very small angle, for example, 0.1° or 1°.

(4) Set the next search direction as the vector between $\mathbf{d}_k^i$ and $\nabla g(\mathbf{u}_k^i)$:

$$\mathbf{d}_{k+1}^i = \frac{\mathbf{d}_k^i}{\left\|\mathbf{d}_k^i\right\|} + \frac{\nabla g(\mathbf{u}_k^i)}{\left\|\nabla g(\mathbf{u}_k^i)\right\|}. \tag{19}$$

(5) Update the current point:

$$\mathbf{u}_{k+1}^i = b^i \frac{\mathbf{d}_{k+1}^i}{\left\|\mathbf{d}_{k+1}^i\right\|} \tag{20}$$

and goto (1).

As shown in Figure 6, the basic idea of the above procedure is to search two points on the β-sphere that satisfy the overlapping condition. If the overlapping condition is not satisfied, we update this point on the β-sphere along the vector with a direction in between the aforementioned two vectors. If a good initial point $\mathbf{u}_0^i$ is chosen, the above iterative process should terminate quickly. Two initial points with opposite directions will be used to search the two *MPP*s on the same β sphere. The selection of initial search points is discussed in Section 3.2.2.

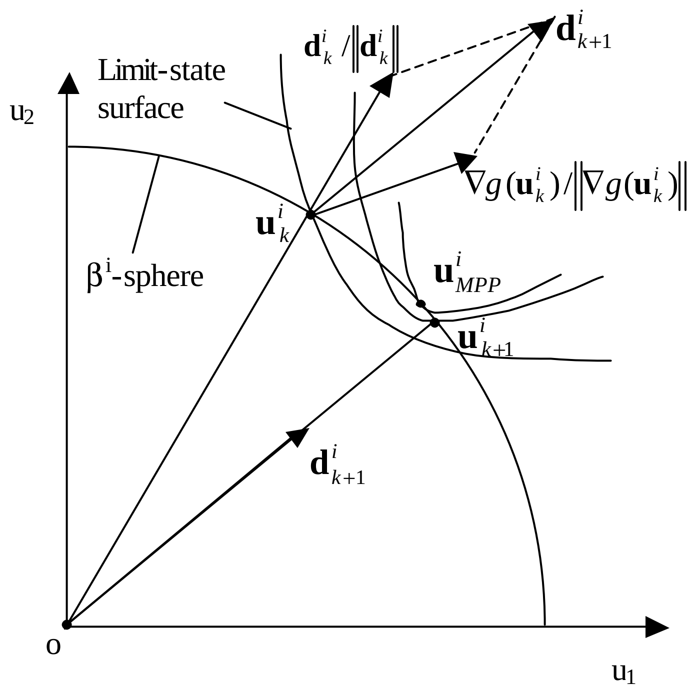



**Figure 6. Searching the *MPP* at $b^{\text{i}}$**

### 3.2.2 Tracing the *MPP* locus and predicting the initial *search point*

In any search that involves an iterative procedure, the starting point becomes critical. In our proposed method, the probability levels are identified incrementally to generate the complete *CDF* of a system output. Utilizing this feature, the *MPPs* identified for the smaller $\beta$s are used to predict the initial search point when searching the *MPP* for a larger $\beta$. The procedure is geometrically demonstrated in Fig. 7 and stated as follows:

(1) For i=1, calculate the gradient $\nabla g(\mathbf{o})$ at the origin and take it as the search direction

$$\mathbf{d}_0^1 = \nabla g(\mathbf{o}). \tag{21}$$

Determine the initial *MPP* search point as:

$$\mathbf{u}_0^1 = b^1 \frac{\mathbf{d}_0^1}{\|\mathbf{d}_0^1\|}. \tag{22}$$

Locate the *MPP* $\mathbf{u}_{MPP}^{1}$ using the procedure presented in Section 3.2.1.

(2) For i=2, increase the radius of β-sphere as

$$b^{2} = b^{1} + \Delta b \ , \tag{23}$$

and take the direction connecting the origin and $\mathbf{u}_{MPP}^{1}$ as the searching direction:

$$\mathbf{d}_0^2 = \mathbf{u}_{MPP}^{1} - \mathbf{u}_{MPP}^{0} , \tag{24}$$

where $\mathbf{u}_{MPP}^{0}$ is the origin.

Determine the initial *MPP* search point as:

$$\mathbf{u}_0^2 = b^{2} \frac{\mathbf{d}_0^2}{\left\| \mathbf{d}_0^2 \right\|} . \tag{25}$$

Locate the *MPP* $\mathbf{u}_{MPP}^{2}$ using the procedure presented in Section 3.2.1..

(3) For i≥3, increase the radius of β-sphere as

$$b^{i} = b^{i-1} + \Delta b \ . \tag{26}$$

Use the three previous *MPP*s to predict the initial search point $\mathbf{u}_0^i$ by a second order polynomial. The coefficients $a_{oj}$, $a_{1j}$ and $a_{2j}$ can be obtained by solving the following equations:

$$\begin{cases} u_{MPP}^{i-1} = a_{oj} + a_{1j} b^{i-1} + a_{2j} (b^{i-1})^2 \\ u_{MPP}^{i-2} = a_{oj} + a_{1j} b^{i-2} + a_{2j} (b^{ii-2}) \\ u_{MPP}^{i-3} = a_{oj} + a_{1j} b^{i-3} + a_{2j} (b^{i-3}) \\ \qquad j = 1, \mathrm{n} \end{cases} \tag{27}$$

The initial search point of the current *MPP* is predicted as

$$u_{0,j}^{i} = a_{oj} + a_{1j} b^{i} + a_{2j} (b^{i})^2 \quad j = 1, \mathrm{n} \, . \tag{28}$$

The predicting curve generated by Eqn. 28 is shown in Fig. 7. The proposed procedure is applied to the search of two *MPP*s on the same β sphere, towards two opposite directions.

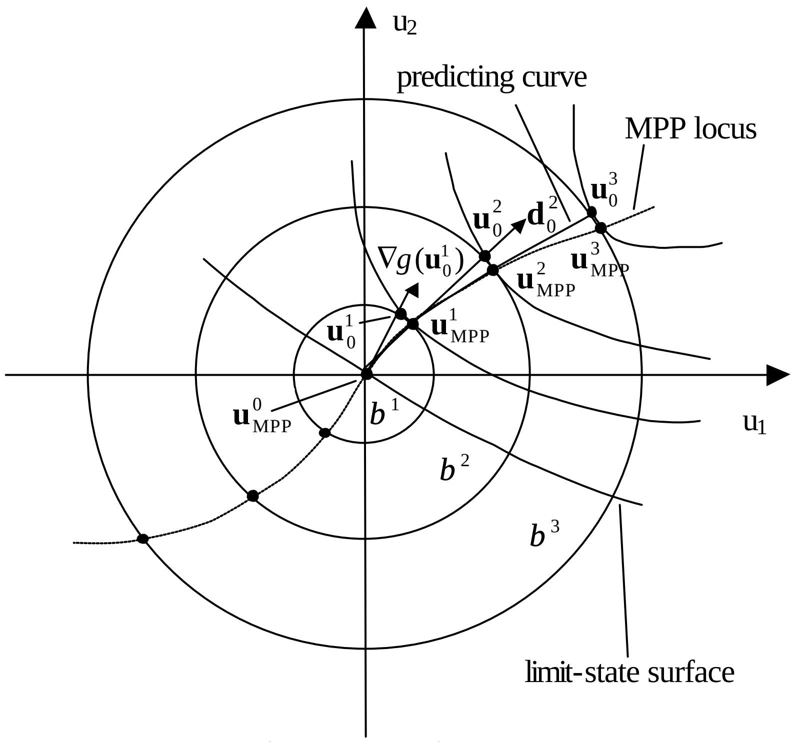



**Figure 7. Tracing *MPP* Locus**

Our proposed method allows us to locate the *MPP*s for small $b$-spheres and then use them to predict the initial search point for the *MPP* on the next $b$-sphere with increased radius. This helps us to locate the right *MPP*s for larger $b$-spheres. The radius of the $b$-spheres are gradually increased and the trace of the *MPPs* is often referred to the *MPP* locus. The procedure of selecting the initial search point (Section 3.2.2) and the *MPP* search algorithm (Section 3.2.1) are integrated, which leads to an efficient and robust *MPP* search procedure.

## 4. EXAMPLES

Two examples are used to demonstrate the effectiveness of the proposed method. Example 1 is a mathematical example and example 2 is a real engineering problem. The *CDF*s and *PDF*s of the system output generated by the proposed method are compared with those obtained from the *MCS* in which very large number of simulations is considered.

### 4.1 Example 1 – A Mathematical Example

Two design variables $\mathbf{x} = \{x_1, x_2\}$ are considered for this example. $x_1$ is normally distributed with mean as 100 and standard deviation as 20; $x_2$ is a lognormal random variable with mean as 15 and standard deviation as 14. The system function is

$$g(\mathbf{x}) = (x_1^3 x_2 - 2x_1 x_2 - x_1^2 - 2x_2 e^{-x_1})/10000 \,. \tag{29}$$

*CDF* values obtained by the *MPPUA* and the *MCS* are compared in Table 1. The number of simulations of *MCS* is $10^7$. We notice that the two sets of *CDF* values are very close to each other. Figs. 8 and 9 provide graphical comparisons of these two *CDF* curves. From Figure 8, we note that the two *CDF* curves obtained by the *MPPUA* and *MCS* almost overlap with each other. From Fig. 9, it is observed that the system output is not symmetrically distributed and this character is well captured by *PDF* curve generated using the proposed method. The two *PDF* curves are also close except there is a slight difference at the pick of the curves. Fig. 10 shows the *MPP* locus in **u** space, which provides us a good visualization of the *MPP* tracing history.

**Table 1. CDF results of example 1**

| y | *CDF* by *MPPDG* | *CDF* by *MCS* | β |
|---|---|---|---|
| 18841.07 | 0.9987 | 0.9988 | 3 |
| 14303.79 | 0.9965 | 0.9968 | 2.7 |
| 10842.49 | 0.9918 | 0.9925 | 2.4 |
| 8204.972 | 0.9821 | 0.9835 | 2.1 |
| 6197.639 | 0.9641 | 0.9664 | 1.8 |
| 4671.936 | 0.9332 | 0.9372 | 1.5 |
| 3513.999 | 0.8849 | 0.8912 | 1.2 |
| 2636.634 | 0.8159 | 0.8248 | 0.9 |
| 1973.093 | 0.7257 | 0.7374 | 0.6 |
| 1472.336 | 0.6179 | 0.6318 | 0.3 |
| 1095.363 | 0.5 | 0.5155 | 0 |
| 811.6728 | 0.3821 | 0.3979 | 0.3 |
| 598.3986 | 0.2743 | 0.2888 | 0.6 |
| 438.6689 | 0.1841 | 0.1962 | 0.9 |
| 319.4687 | 0.1151 | 0.1244 | 1.2 |
| 230.8365 | 0.0668 | 0.0734 | 1.5 |
| 165.1907 | 0.0359 | 0.0402 | 1.8 |
| 116.7804 | 0.0179 | 0.0204 | 2.1 |
| 81.2578 | 0.0082 | 0.0096 | 2.4 |
| 55.345 | 0.0035 | 0.0041 | 2.7 |
| 36.58 | 0.0013 | 0.0016 | 3 |

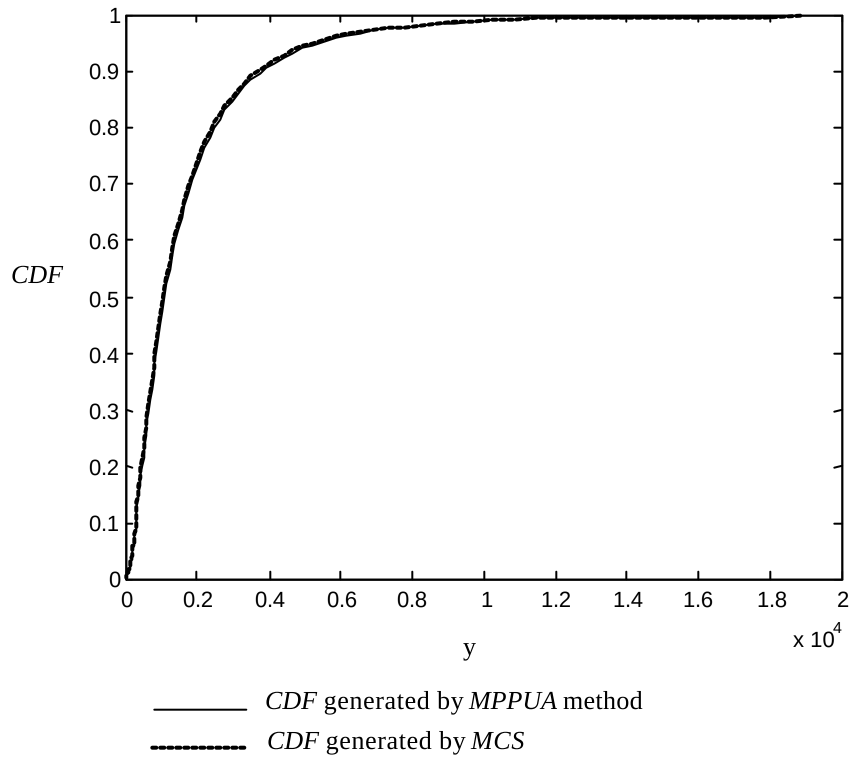



**Figure 8. Example 1 – *CDF***

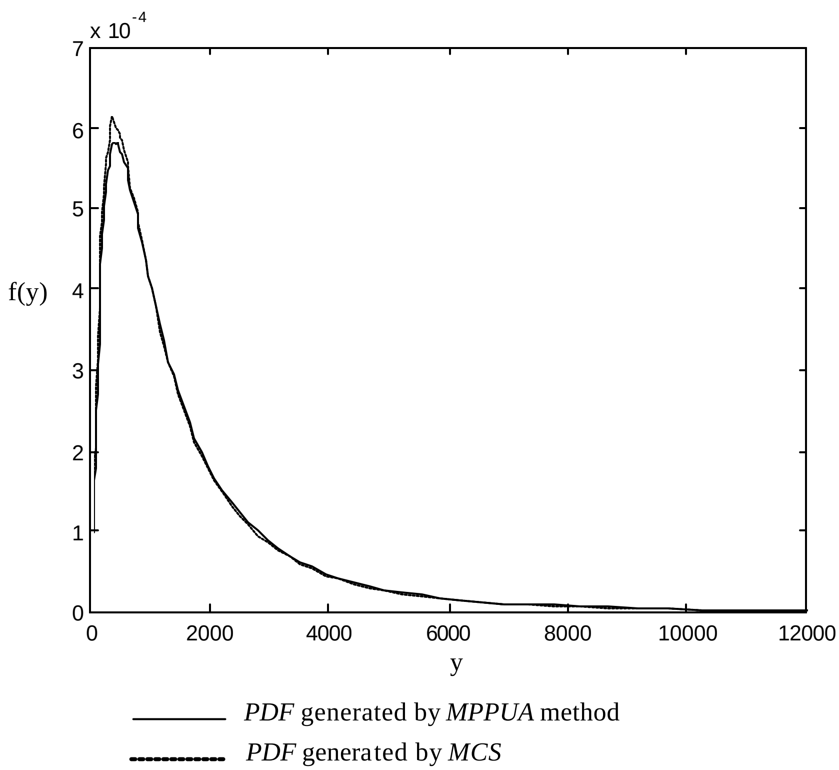



**Figure 9. Example 1 – *PDF***

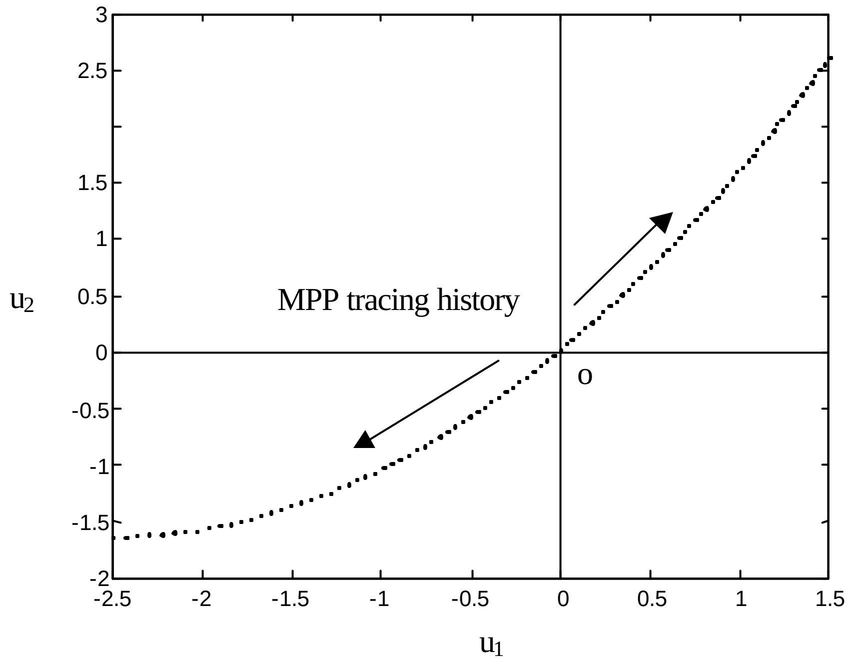



**Figure 10. Example 1 – *MPP* Locus in u Space**

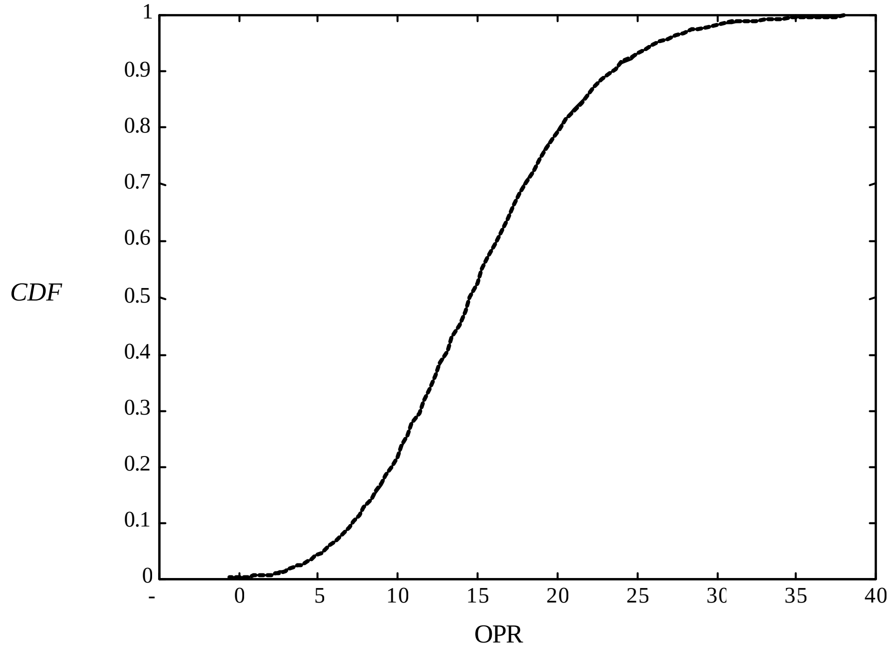
### 4.2 Example 2 – Pratt & Whitney (PW) Engine Design

The PW engine design problem is used in this study to illustrate the applicability of the proposed *MPPUA* for problems that are in a complex domain and taking a lot of computational resources. The problem statement of the PW engine design problem is provided in [35]. A total of eight continuous design variables are considered. They are the Fan Pressure Ratio (FPR), the Exhaust Jet Velocity Ratio (VJR), the Turbine Inlet Temperature or the Combustor Exit temperature (CET), the High Compressor Pressure Ratio (HPCPR), the Low Compressor Pressure Ratio (LPCPR), High Turbine Compressor Efficiency (EHPC), High Turbine Efficiency (EHPT), and Low Turbine Efficiency (ELPT). The distributions of these design variables are listed in Table 2.

**Table 2 Distributions of design variables in engine design**

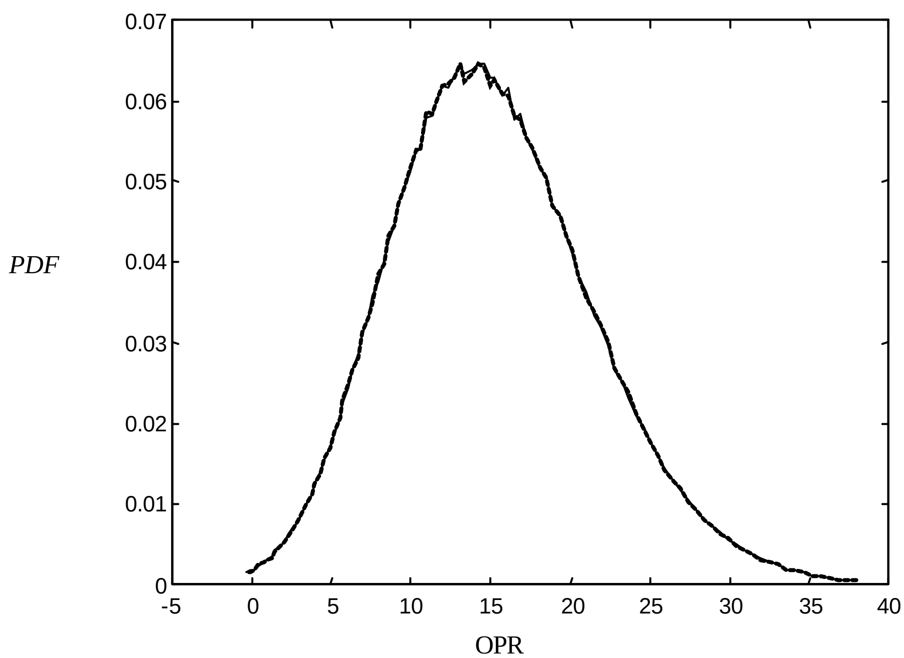
| | Mean | Standard Deviation | Distribution type |
|---|---|---|---|
| FPR | 0.8 | 0.1 | Lognormal |
| VJR | 2600 | 300 | Lognormal |
| CET | 15 | 1.5 | Lognormal |
| HPCPR | 3 | 0.3 | Lognormal |
| LPCPR | 1.45 | 0.1 | Weibull |
| EHPC | 0.89 | 0.08 | Weibull |
| EHPT | 0.93 | 0.01 | Weibull |
| ELPT | 0.805 | 0.01 | Weibull |

The system outputs are Overall Pressure Ratio (OPR), Fan Diameter (FANDIA) and HPTPR (High Turbine Pressure Ratio), all of which are nonlinear and complicated functions evaluated by a thermal analysis program called SOAPP (Varadarajan, et al., 2000). Figs. 11 and 12 indicate that the *MPPUA* method generates excellent estimations of *CDF* and *PDF* for OPR when the results from *MCS* are taken as a reference (simulation number is $10^7$). The *MPPUA* method also generates good estimations of *CDF* and *PDF* for FANDIA even through there exist some slight differences (see Figs. 13 and 14). However for HPTPR, the estimated *CDF* and *PDF* curves are slightly shifted to the right of those obtained from *MCS* (see Figs. 15 and 16). This indicates that the model function HPTPR may have large curvatures at *MPP*s, and second-order probability estimate can be used to improve the accuracy. The total number of the model function evaluations is 398 for this example. If the same number is taken for *MCS*, the results from *MCS* will be much less accurate than that from the proposed approach.




CDF generated by *MPPUA* method
CDF generated by Monte Carlo Simulation

**Figure 11. Example 2 – *CDF* of OPR**




PDF generated by *MPPUA* method
PDF generated by *MCS*

**Figure 12. Example 2 – *PDF* of OPR**

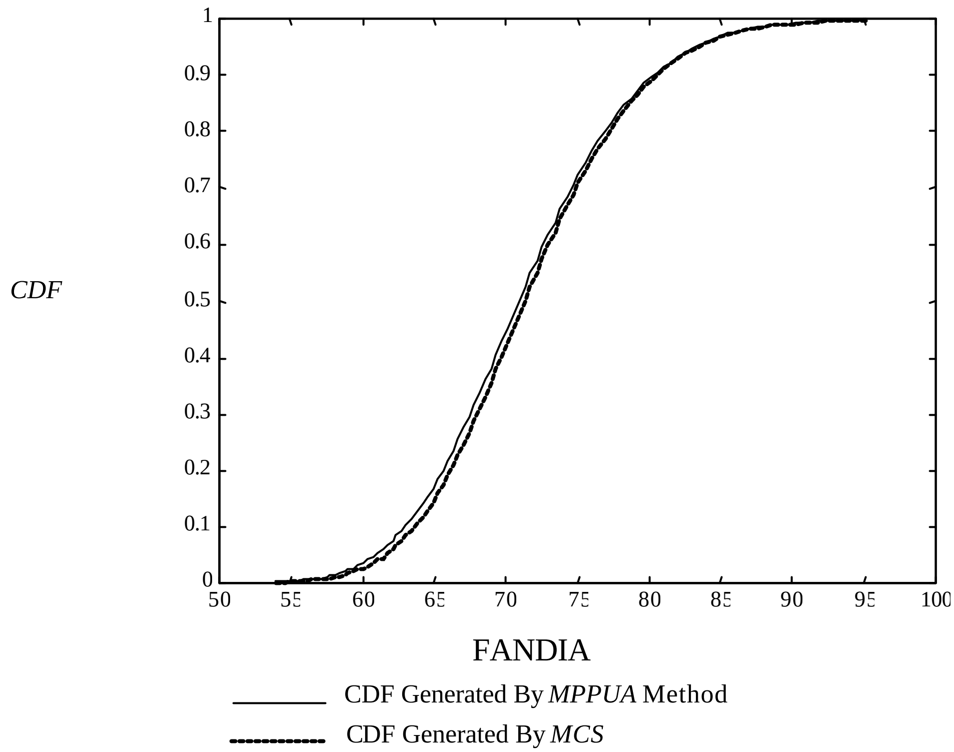



**Figure 13. Example 2 – *CDF* of FANDIA**

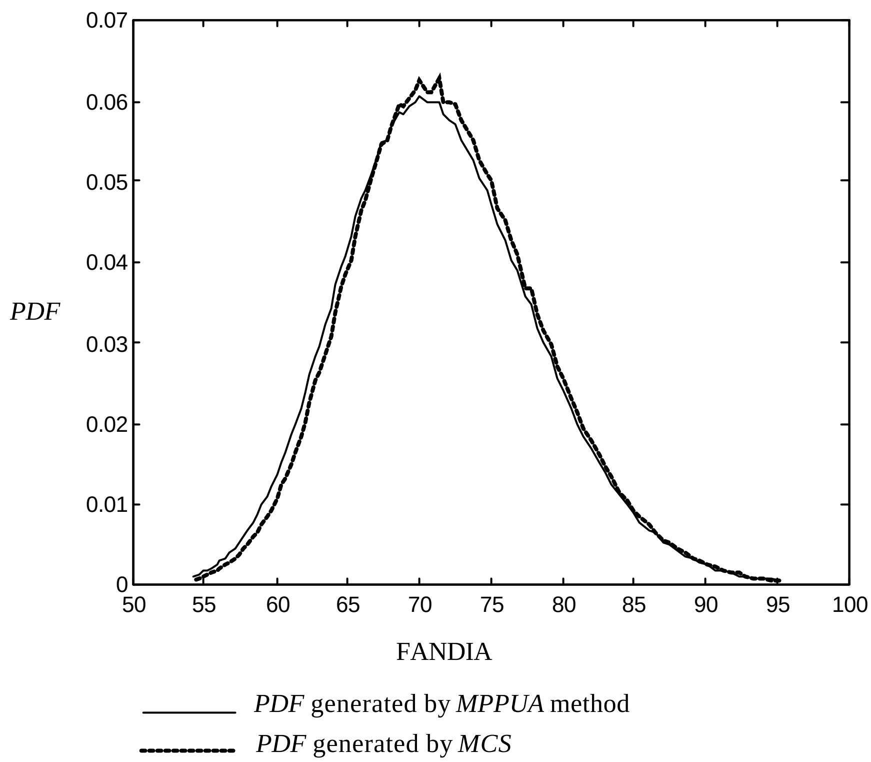



**Figure 14. Example 2 – *PDF* of FANDIA**

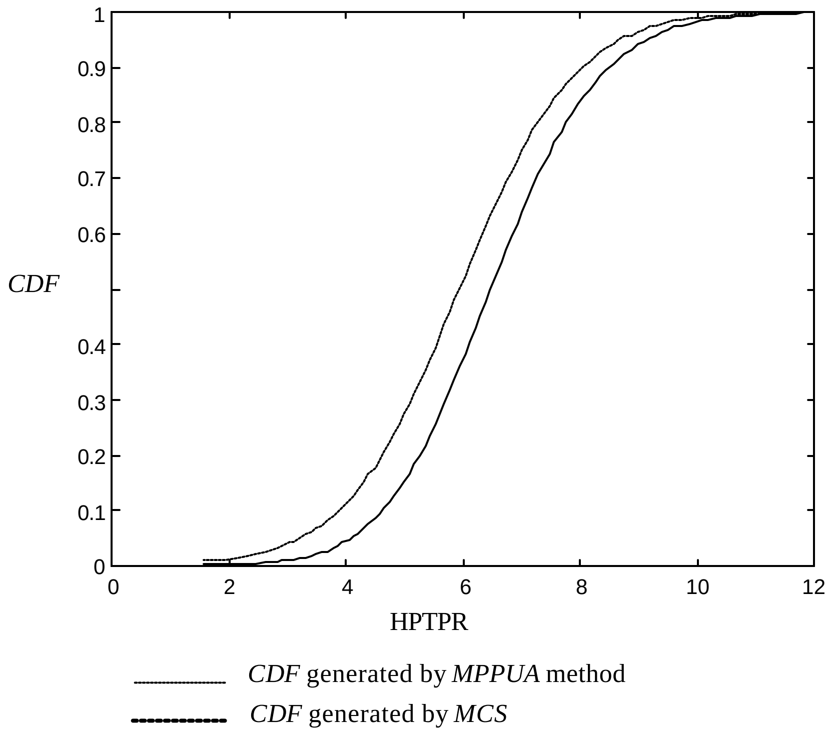



**Figure 15. Example 2 – *CDF* of HPTPR**

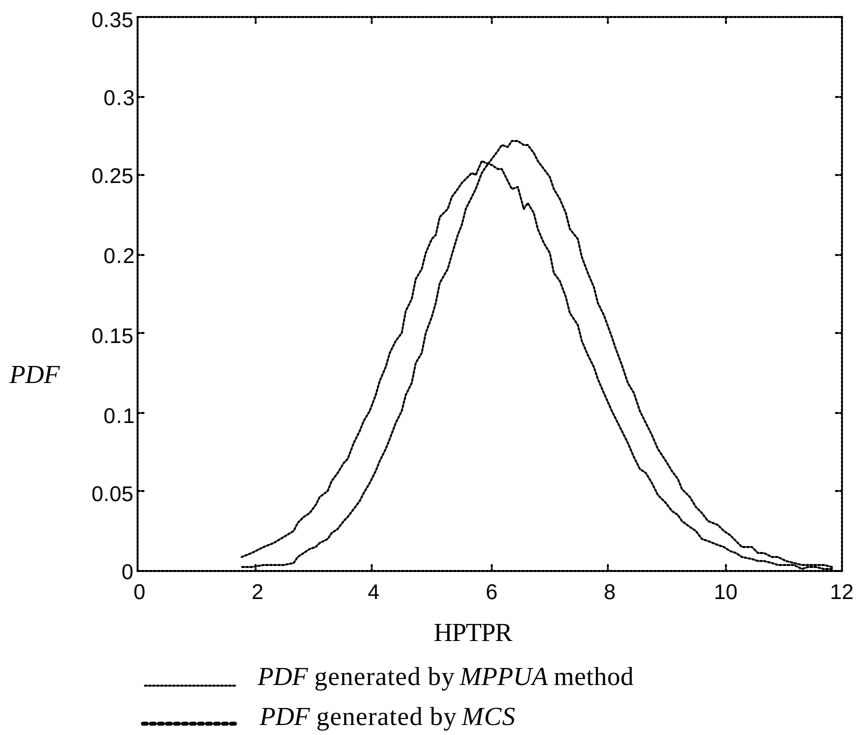



**Figure 16. Example 2 – *PDF* of HPTPR**

### 4.3 Discussions

Based on the two examples, we note that the *MPPUA* method provides us very good estimations of the probabilistic distribution of a system output. We comment on its performance in the following several categories:

1) Efficiency: The efficiency of the proposed method depends on the speed of *MPP* search. Only two or three iterations are needed to locate the *MPP* at each $\beta$ for the above two examples. Our proposed search strategy (section 3.2.1) and the method of selecting the initial search point along the *MPP* locus (section 3.2.2) have significantly enhanced the performance of utilizing the *MPP* concept.
2) Robustness: Our proposed search algorithm in Section 3.2.1 will work for functions with different curvatures. Tracing the *MPP* locus is very helpful for the convergence of *MPP* search when dealing with different types of system behavior because it always gives us good initial search points to start with. As discussed in section 3.2.2, we start *MPP* search on the smallest $b$-sphere. It is easier to find a correct *MPP* on a small $b$-sphere compared with a large $b$-sphere. The initial point for the *MPP* search on the larger $b$-sphere is predicted using the *MPPs* for smaller $b$-spheres. Good starting points help to converge to the right *MPP* points in an efficient manner.

In spite of the many advantages listed, we should keep in mind that our proposed method is only an approximation method for predicting the probabilistic distributions in a certain situations. Its accuracy depends on many issues such as the linearity of the transformed system function at the *MPP*, the normalization of the non-normal random variables, the stopping criteria of the *MPP* search, etc. The linearity is the key factor that affects the accuracy. Highly nonlinear model function (large magnitude of the principal curvatures at *MPP* in **u** space) may result in significant errors of *CDF* estimation. To improve the accuracy under this situation, the second-order approximation to the probability $P\{g(\mathbf{x}) < y_i\}$ can be used to fit the model function at the *MPP*. Usually a paraboloid or a sphere function [36,37] is used. The simplest of these methods, based on a paraboloid fitting, is the asymptotic formulation [32]:

$$P[g(\mathbf{x}) \leq y_i] = \Phi(-b^i)\prod_{j=1}^{n-1}(1 + b^i r_j)^{-1/2} . \tag{30}$$

where $r_j$ denote the main curvatures of the model function or limit-state function at the *MPP*. When there exist more than one local minimal distance points (multiple *MPP*s), an approximation of the solution can be obtained by fitting planes or paraboloid surfaces at all these points [38]. Obviously, the improvement of accuracy will involve the evaluations of second derivatives of a function. To avoid this, a *MPP* based importance sampling method can be used. This method is implemented by sampling around the *MPP* rather than over the whole random space. Therefore the number of simulation is much less than that of the general *MCS*. This method is fully documented in [13].

While the accuracy of the *MPPUA* method mainly depends on the features of a system model, its efficiency relies on the problem dimension (the number of random variables). Since the *MPPUA* method is a gradient based method, the number of function evaluations will be approximately equal to the number of random design variables times the number of derivative updates. Therefore, when the number of random variables is very large (for example, bigger than 50), the advantage of the *MPPUA* method over simulation methods may diminish. In this situation, *MCS* may be more efficient to use than the *MPPUA* method. To deal with large-scale problems, one solution is to use the design of experiments techniques to screen out unimportant variables [39] and then to implement the *MPPUA* method in the reduced design space.

## 5. CLOSURE

Adopting probabilistic model in engineering design is very important for designers to make intelligent and reliable decisions under uncertainty. Uncertainty analysis is required to propagate the effect of uncertainty on a system output. The research of this paper is focused on how to capture the complete probabilistic distribution of a system output with the existence of uncertainty in system input. To accomplish this goal, a *MPP* based uncertainty analysis method is proposed. The method utilizes the concept of *MPP* to generate the cumulative distribution function (*CDF*) of a system output by evaluating probability estimates at a serial of limit states across a range of output performance. The probability density function (*PDF*) can then be estimated by the differential of *CDF*. A novel *MPP* search algorithm is presented that employs a set of searching strategies, including adjusting search directions based on derivatives, tracing the *MPP* locus, and predicting the initial point for *MPP* search.

With the capability of generating the probabilistic distribution of a system output, the proposed method can serve as a useful tool in decision making under uncertainty. In addition to utility function optimization, it can also be used in many forms of probabilistic design, such as robust design, stochastic optimization, and reliability based design. The proposed method can be used to assist the evaluation of mean and variance (both are major concerns in robust design), the probability of success (if the system output is considered as the objective in optimization, or the limit-state in reliability analysis), and the feasibility robustness (if the system output is considered as a constraint). It is expected that the proposed method will generally provide much better accuracy compared to sensitivity-based approximations and response surface modeling approaches, and much better efficiency compared to sampling-based methods. Our future study will be towards a comprehensive comparative study of different methods of uncertainty analysis for a variety of problems with different dimensions. Considering the different usages of *MPPUA* in engineering design applications, we will test the accuracy and efficiency of different methods against different metrics such as the evaluation of expected utility, the mean and standard deviation, and the limit state evaluation at the tails of distributions. The results from comparative studies will serve as useful guidelines for choosing the most efficient technique under different circumstances.

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