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A MOST PROBABLE POINT BASED METHOD FOR UNCERTAINTY ANALYSIS

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ABSTRACT

Uncertainty is inevitable at every stage of the life cycle development of a product. To make use of probabilistic information and to make reliable decisions by incorporating decision maker's risk attitude under uncertainty, methods for propagating the effect of uncertainty are therefore needed. When designing complex systems, the efficiency of methods for uncertainty analysis becomes critical. In this paper, a most probable point (*MPP*) based uncertainty analysis (*MPPUA*) method is proposed. The concept of the *MPP* is utilized to generate the cumulative distribution function (*CDF*) of a system output by evaluating the probability estimates at a serial of limit states. To improve the efficiency of locating the *MPP*, a novel *MPP* search algorithm is presented that employs a set of searching strategies, including evaluating derivatives to direct a search, tracing the *MPP* locus, and predicting the initial point for *MPP* search. A mathematical example and the Pratt & Whitney (PW) engine design are used to verify the effectiveness of the proposed method. With the *MPPUA* method, the probabilistic distribution of a system output can be generated across the whole range of its performance.

1. INTRODUCTION

The notion that engineering design is a decision-making process has been paid more and more attention in recent years (De Neufville, 1990; Hazelrigg, 1996, 1998, 1999; Pinto, 1998). To rank order different design options, designers (decisions makers) usually rely on mathematical models (or computer simulation models) to predict system behaviors and ultimately design utility. However, despite the enormous power of computational models, they are not perfect because all of them are only abstractions of the realities. Due to the lack of knowledge and the use of assumptions by model builders, uncertainty is inevitable for models at every stage of s life cycle. As the result, the model-predicted performance of a system and the actual system performance will deviate at s certain level.

When uncertainty is considered in engineering design, probabilistic design models are adopted instead of deterministic models. Unfortunately, probabilistic models require order(s) of magnitude increase in the analysis in comparison to the deterministic models (DeLaurentis and Marvris, 2000). The problem is worse when a design model involves much more complex computations, such as calculations of stresses, temperatures, heat transfer rates, and fluid flows through numerical algorithms or simulation tools that could involve finite element analysis, computational fluid dynamics, etc. One of the challenges to use probabilistic design models is to capture the effect of uncertainty on a system output in an efficient manner. The problem can be stated as: *given the probability distributions of the random variables* in a system (e.g., those of the *design variables and parameters*), *what should be the probability distribution of a system output*? The issue is how to propagate the effect of uncertainty. This process is often referred to as uncertainty analysis. Our aim in this paper is to develop efficient methods for uncertainty analysis that are usable in engineering design applications.

Once the effect of uncertainty is propagated, the information on the distribution of a system output will be used further for making reliable decisions through optimization. The decision-based design (DBD) (Hazerlrigg 1998) is such a framework that provides the capability to make use of probabilistic information and to incorporate decision maker's risk attitude under uncertainty through the utility function optimization.

A detailed review of the related work on uncertainty analysis is provided in Section 2.1. It is found that most of the exiting works only focus on generating the mean and the variance of a system output or the range of deviation. This is often not sufficient to describe the whole probabilistic distribution of a system output, and most of the time we do not know in advance what type of distribution the system output will follow. The methods available for generating complete probabilistic distributions are often expensive to use and are not considered as usable tools for real engineering applications. In this paper, we propose an efficient method uncertainty analysis based on the concept of the most probable point (*MPP*) that was originated from the field of structural reliability. We call this method the most probable point based uncertainty analysis (*MPPUA*) method. To improve the efficiency of locating a *MPP*, a novel *MPP* search algorithm is presented that employs a set of searching strategies, including evaluating derivatives to direct a search, tracing the *MPP* locus, and predicting the initial point for *MPP* search.

This paper is organized as follows. In Section 2, the frame of references of our research is presented. The concepts of the most probable point (*MPP*) are introduced. The proposed *MPPUA* method is presented in Section 3 along with the discussions of an *MPP* search algorithm and other search strategies. In Section 4, two examples are used to demonstrate the effectiveness of the proposed method. The accuracy and efficiency of the *MPPUA* method are examined. Section 5 is the closure of this paper.

2. RELATED WORK AND OUR TECHNOLOGY BASE

2.1 Review of Related Work

Researches on the sources of uncertainty and how to model them can be found in the literature (the recent references including Manners, 1990; Laskey, 1996; Ayyub and Chao, 1997; Du and Chen, 1999a). The commonly used method for uncertainty analysis is the sensitivity based approximation approach that includes the worst case analysis and the moment matching method (Eggert, 1991; Parkinson, et al., 1993, Chen, et al., 1996; Du and Chen, 1999b, Du and Chen, 2000, and Du, et al., 2000). With the worst case analysis, all the fluctuations are assumed to occur simultaneously in the worst possible combinations and based on this assumption, the worst value of the system output (extreme condition) can be found by the first order Taylor expansion or optimization. With the moment matching method, the first order moment (mean value) and the second order moment (standard deviation) of a system output are obtained.

Reliability analysis is another class of methods that focus on evaluating the probability of the event that a system output is less or bigger than a pre-specified value. The typical methods in the filed of reliability analysis include FORM (first order second moment method) and SORM (second order reliability method) (Melchers, 1999).

Compared with the preceding methods, Monte Carlo Simulation (*MCS*) is a more comprehensive method that can generate the cumulative distribution function (*CDF*) and the probability density function (*PDF*) of a system output based on data sampling. The shortcoming of *MCS* is that great computational effort is required for any general cases. Some modified *MCS* methods have been proposed to improve the computational efficiency. Among them are the importance sampling method (Kahn, 1956; Rubinstein, 1981), the Latin hypercube sampling method (Walker, 1986), the shooting Monte Carlo approach (Brown and Sepulveda, 1997), and the directional simulation (Ditlevsen, et al., 1987). However, even with modifications, data sampling techniques are generally not affordable in the design of complex engineering systems.

To reduce the computational effort, surrogate models that employ polynomial chaos expansion are proposed by some researchers (Isukapalli and Georgopoulos, 1998; Isukapalli, 1999; Wang, 1999). In Isukapalli's work, random input variables are transformed into standard normal distributed variables and then the response surface model of the original model is created in the form of the Hermite-polynomial. Wang used the more general polynomial chaos expansion, which also includes the Hermite-polynomial, to build the surrogate models. After the surrogate models are established, *MCS*, design of experiments (DOE) (Box, 1978), or other methods are employed to generate the distribution of a system output based on the inexpensive surrogate models. It should be noted that these surrogate models are generated only in the neighborhood of a candidate design point rather than over the whole design space. When the number of design options (from which the preferred design will be chosen) is large, a great amount work is required to create the surrogate models and the computational burden may still not be affordable. The alternative way is to directly create the response surface models as the functions of random variables and design variables over the entire design space (Sues, et al., 1995; Kowal and Mahadevan, 1998; Koch, et al., 1999; Mavris, et al., 1999). Recently, considerations of both the uncertainty of input variables and the uncertainty of model itself (model error) have been reported for multidisciplinary design (Gu and Renaud, 1998; Du and Chen, 1999a).

As mentioned in Section 1, most of the existing works on uncertainty analysis only focus on generating the mean and the variance of a system output or the range of the performance deviation. There is a need for developing efficient uncertainty analysis methods that can be used in the design of complex system to derive full probability distribution functions.

2.2 Our Technology Base

A generic uncertainty analysis problem is considered in this work. We let model inputs $\mathbf{x} = \{x_1, x_2 . \cdots x_n\}$ be mutually independent random variables with the cumulative distribution function (*CDF*) $F_i(x_i)$ and the probability density function (*PDF*) $f_i(x_i)$ ($i = 1, n$). The system model $g(\mathbf{x})$ maps model inputs $\mathbf{x} = \{x_1, x_2, \cdots x_n\}$ into an output y as:

$$y = g(\mathbf{x}). \tag{1}$$

Theoretical methods can be used to generate the *PDF* of the output *y* (Papoulis, 1991). However they are difficult or even impossible to use for complex system behaviors. Simulation methods (for example, Monte Carlo method) are used whenever the analytical methods are not practical. But usually, they are not efficient. Alternatively, analytical methods, such as the *MPP* method, can be used.

The *MPP* method was originally developed in the field of reliability analysis (Hasofer and Lind, 1974). It requires that limit-state functions be defined and then the probability of the limit-state functions bigger or less than zero can be evaluated approximately. The limit-state function is defined as:

$$z(\mathbf{x}) = y - c = g(\mathbf{x}) - c, \tag{2}$$

where *c* is a constant.

The *MPP* is formally defined in a coordinate system of an independent and standardized normal vector $\mathbf{u} = \{u_1, u_2, \cdots u_n\}$. The input variables $\mathbf{x} = \{x_1, x_2, \cdots x_n\}$ (in the original design space) are transformed into the standard normal space $\mathbf{u} = \{u_1, u_2, \cdots u_n\}$. The most commonly used transformation is given by Rosenblatt (Rosenblatt, 1952) as

$$u_i = \Phi^{-1}[F_i(x_i)] \quad (i = 1, n), \tag{3}$$

where Φ^{-1} is the inverse of a normal distribution function. Eqn. 3 stands for that the transformation maintains the *CDF*s being identical both in **x** space and **u** space (see Figure 1).

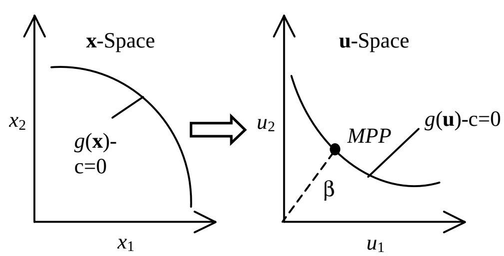


Figure 1. The Transformation of input variables

The limit-state function is now rewritten as

$$z(\mathbf{u}) = g(\mathbf{u}) - c. \tag{4}$$

Hasofer and Lind (1974) defined β as the shortest distance from the origin to a point on the limit-state surface in **u** space (see Fig. 1). Mathematically, it is a minimization problem with an equality constraint:

$$\beta = \min_{\mathbf{u}} \|\mathbf{u}\| \tag{5}$$

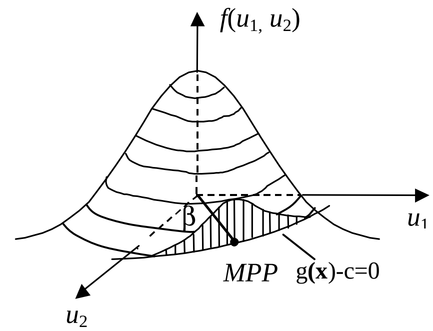
subject to

$$g(\mathbf{u}) - c = 0. \tag{6}$$

The solution $\mathbf{u}_{MPP}$ of this minimization problem is called the most probable point (MPP).

From Fig. 2, we see that the joint probability density function on the limit-state surface has its highest value at the *MPP* and so *the MPP has the property that in the standard normal space it has the highest probability of producing the value of limit-state function z*(**u**) (Wu, 1990). β is also referred to as the safety index in reliability analysis and the MPP becomes the critical design point.

Figure 2. The *MPP* concept



If the limit-state function $z(\mathbf{u})$ is linear, the accurate probability estimate at the limit state is given by the equation:

$$P\{z(\mathbf{u}) < 0\} = P\{g(\mathbf{x}) < c\} = \begin{cases} \Phi(\beta) & \text{if } P \ge 0.5 \\ 1 - \Phi(\beta) & \text{if } P < 0.5 \end{cases} \tag{7}$$

The above equation provides an easy correspondence between the probability estimate and the safety index. If $z(\mathbf{u})$ is nonlinear, a good approximation can still be obtained by the above equation, provided that the magnitude of the principal curvatures of the limit-state surface in the **u** space at the *MPP* is not too large (Mitteau, 1999). If the limit-state function is highly nonlinear, an alternative second-order approximation at the *MPP* can be used, which takes into account the curvature of the limit-state surface around the *MPP* (Breitung, 1984, Tvedt, 1990).

3. A MPP BASED METHOD FOR UNCERTAINTY ANALYSIS

In this work, a most probable point (*MPP*) based uncertainty analysis (*MPPUA*) method is developed to estimate the probabilistic distribution of a system output. The concept of the *MPP* is utilized to generate the cumulative distribution function (*CDF*) of the system output by output by evaluating the probability estimates at a serial of limit states. A novel *MPP* search algorithm is also developed to support the procedure of *CDF* generation.

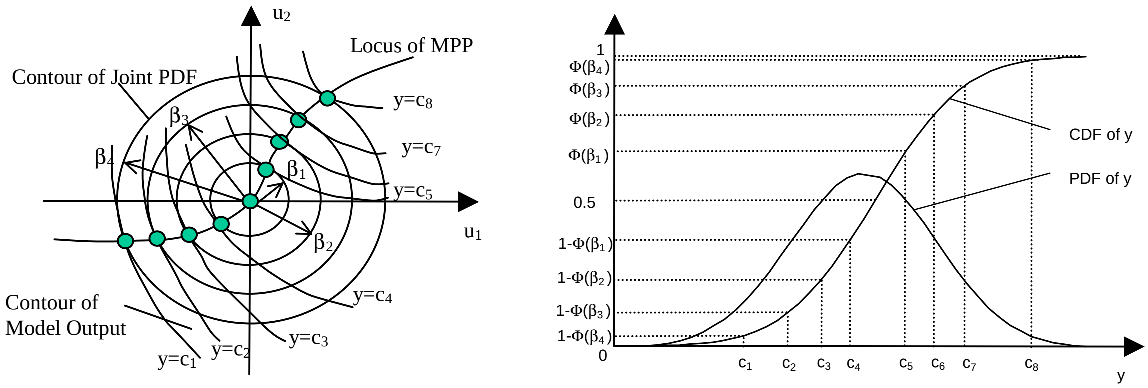


Figure 3. Concept of the *MPPUA* Method

3.1 Generating probabilistic distribution of a system output

The basic idea of the *MPPUA* method is to make use of the property of the *MPP* to approximate the probability output at a serial of limit states. As we discussed in the previous Section 2.2, *MPP* has the property that in the standard normal space it has the highest probability of producing the value of a limit-state function, and the information of the shortest distance can be used to predict the probability of limit-state function less than zero. This is very relevant to the definition of the *CDF* of a system output , i.e.,

$$F_y(c) = P\{y < c\} = P\{g(\mathbf{u}) < c\}\ , \qquad (8)$$

which is equivalent to

$$F_y(c) = P\{g(\mathbf{u}) - c \le 0\} = P\{z(\mathbf{u}) < 0\}\ . \qquad (9)$$

The analogy is that if we could identify the *MPP* for a system output y (or g(**u**)) at a limit state c, this will provide us an estimate of the *CDF* of y when $y = c$. For each *MPP*, there is a corresponding value of β. Based on Eqn. 9, the *CDF* of y at $y = c$ becomes

$$F_y(c) = P\{y < c\} = P\{z(\mathbf{u}) < 0\} = \begin{cases} \Phi(\beta) & \text{if } P \ge 0.5 \\ 1 - \Phi(\beta) & \text{if } P < 0.5 \end{cases} .(10)$$

If we could evaluate the *CDF* values using this way for a set of c values chosen in a range of system output y, then we will be able to obtain a (discretized) complete *CDF* across a range of y. Since the reasonable range of a system output is generally not known in advance, difficulty will arise when choosing the set of c values (limit states) because its values largely depend on an application. However, there is a direct correspondence between c and β, where the latter can be easily assigned a fixed range, for example, [0, 3] or [0, 4], which correspond to *CDF* values in the range of [0.0013, 0.9987] and $[3.1671\times10^{-5}, 0.99997]$ (close enough to [0 1]) based on Eqn. 10, respectively. To locate a set of *MPPs* at these points, we choose to discretize the corresponding β values into equal segments. A serial of β *is* chosen as*:*

$$\beta^i = \frac{\beta_{\max}}{M}\ (i = 1, M)\ , \qquad (11)$$

where M is the number of points picked.

Geometrically as shown in Figure 3, in the standard normal space **u**, we will consider a family of concentric spheres with radius β^i and we call these spheres β-spheres. Corresponding to each β, the MPP is identified. The limit state values c and the probability estimates F_{yi} are then determined. The right side picture in Figure 3 illustrates the determination of the full cumulative density function based on the evaluations of a set of *MPP*.

Since every β^i yields two values of F_{yi} according to Eqn. 10 and the origin ($\beta = 0$) in the standard normal space **u** gives a *CDF* value 0.5, the total number of F_{yi} and that of β^i have the following relationship

$$K = 2M + 1\ , \qquad (12)$$

where K is the total number of F_{yi} .

Note that the *MPP*s are identified in the standard normal space **u** and they should be transformed into the initial **x** space. Once the **x** values of the *MPP* are identified, system output y_i (same as the limit-state value c) can then be evaluated using the system function g(**x**). When evaluating the values of *CDF*, for the *MPP* with the larger system output, we use the upper expression of Eqn. 10; and for the *MPP* with the smaller output value, the lower expression should be used.

Based on the proceeding discussions, we summarize the procedure of generating the *CDF* of a system output y as follows:

(1) Specify a set of safety index β^i ($i = 1, M$) in the range [0, 3] or [0, 4];
(2) Calculate the corresponding *CDF* values using Eqn. 10.
(3) For a given β^i, find *MPP* $\mathbf{u}^i_{MPP}$ in **u** space (details see Section 3.2);
(4) Map $\mathbf{u}^i_{MPP}$ into **x** space to obtain $\mathbf{x}^i_{MPP}$ using

$$x^i_{MPP,j} = F_i^{-1}[\Phi(u^i_{MPP,j})] \quad (i = 1, K;\ j = 1, n). \qquad (13)$$

(5) Evaluate the value of a system output at a *MPP by*

$$y_i = F(\mathbf{x}^i_{MPP}) \quad . \qquad (14)$$

Once the discretized *CDF* is obtained, we can then obtain the *PDF*: $(f_y(y_i), y_i)$ using

$$f_y(y_i) = \frac{F_y(y_i) - F_y(y_{i-1})}{y_i - y_{i-1}} \tag{15}$$

3.2 Searching the MPP

From the above procedure, we see that the key to generate the correct probabilistic distribution function is to find the right *MPP* for different β^i. Various techniques have been proposed in the literature to search the *MPP*, such as the hypersphere method (Ticky, 1993), the directional cosines method (Ang and Tang, 1984), the advanced mean value (*AMV*) (Wu, et al., 1990), the sampling-based *MPP* search method (Wu, 1998), and the optimization method. Unfortunately, there has been no single search algorithm that could succeed to find the right *MPP* for all situations. As stated by Tichy (Tichy, 1993), the common problems are: 1) The *MPP* obtained appears logical, but it is incorrect, as the iteration leads to a local extreme of the transformed limit-state functions; 2) Solution does not converge. For example, the search process may oscillate between two points. The newly developed sampling based *MPP* searching algorithm (Wu, 1998) seems to be highly robust, but it needs many iterations and much sampling to obtain the correct *MPP*.

In this paper, we develop a new algorithm that uses the derivative information of the limit-state function to direct the search and traces the *MPP* locus step by step through the entire *MPP* search process when generating the complete *CDF*. The algorithm is expected to be efficient and robust as it suits the nature of our proposed procedure in which probability levels are identified incrementally. Two issues are involved in the proposed algorithm. One is the *MPP* search algorithm for a specific β^i and the other is the determination of the initial point for each search. We will discuss them in detail next.

3.2.1 Searching the *MPP* at β^i

Graphically speaking, *MPP* is the tangent point of the hypersphere (called β-sphere with radius β^i) and the limit-state surface in the **u** space (see Fig. 4). The *MPP* search becomes to locate this tangent point, where the vector $\mathbf{d}^i$ connecting the *MPP* $\mathbf{u}^i_{MPP}$ and the origin **o** should overlap with the gradient $\nabla g(\mathbf{u}^i_{MPP})$ of the function $g(\mathbf{u})$. In other words, the angle of $\mathbf{d}^i$ and $\nabla g(\mathbf{u}^i_{MPP})$ should be zero, expressed as

$$\alpha^i = \cos^{-1}\frac{\mathbf{d}^i \bullet \nabla g(\mathbf{u}^i_{MPP})}{\left\|\mathbf{d}^i\right\| \cdot \left\|\nabla g(\mathbf{u}^i_{MPP})\right\|} = 0, \tag{16}$$

where $\nabla g(\mathbf{u}^i_{MPP}) = \left\{\frac{\partial g(\mathbf{u})}{u_1}, \frac{\partial g(\mathbf{u})}{u_2}, \cdots, \frac{\partial g(\mathbf{u})}{u_n}\right\}_{\mathbf{u}^i_{MPP}}$. (17)

The procedure that we developed to locate a *MPP* on the β^i sphere is as follows:

At the current point on β^i sphere, calculate the gradient ∇g of the limit-state function and the vector **d** that connects the origin and the current point. If the angle between ∇g and **d** is smaller than the stopping criterion (for instance, 0.1 degrees), we accept the current point as an *MPP*. If the angle is larger, a new search direction which is between ∇g and **d** and on the plane of ∇g and **d** is chosen. The intersection of this new direction and the β^i sphere then becomes our updated point (new current point). Repeating the procedure until the angle reduces to the accepted value. If a good initial point is chosen, this iterative process will converge quickly. This will be addressed in the next section.

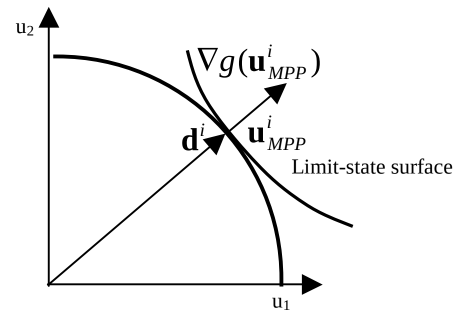


Figure 4. Overlapping Condition at *MPP*

3.2.2 Tracing the *MPP* locus and predicting the initial search point

A good starting point of *MPP s*earch will lead towards the right point in an efficient manner. We employ the strategy of tracing *MPP* locus by increasing the radius of β-sphere gradually from 0 to its upper limit 3 or 4. In our proposed method, the probability levels are identified incrementally to generate the complete *CDF* of a system output. Utilizing this feature, the *MPPs* identified for the smaller βs are used to predict the initial search point when searching the *MPP* for a larger β. The procedure is geometrically demonstrated in Fig. 5. The origin o is the first *MPP* $\mathbf{u}^0_{MPP}$. Along the gradient $\nabla g(\mathbf{u}^0_{MPP})$, a point $\mathbf{u}^1_0$ on the β^1 is found, which is used as the initial point for searching *MPP* $\mathbf{u}^1_{MPP}$ following the procedure developed in previous section. After the two *MPP*s $\mathbf{u}^0_{MPP}$ and $\mathbf{u}^1_{MPP}$ are available, we use the direction from $\mathbf{u}^0_{MPP}$ to $\mathbf{u}^1_{MPP}$ to predict the initial point $\mathbf{u}^2_0$ on the next β^2 sphere. Once the *MPP* $\mathbf{u}^2_{MPP}$ is obtained, we then use the three *MPP*s $\mathbf{u}^0_{MPP}$, $\mathbf{u}^1_{MPP}$ and $\mathbf{u}^2_{MPP}$ to predict the next *MPP*. For three successive *MPP*s, a quadratic curve can be used to approximate the *MPP* locus, which is shown in Fig. 5 entitled by "predicting curve". The common point of the predicting curve and β^3 sphere is the predicted initial point for $\mathbf{u}^3_{MPP}$. Analogously, the initial points on the next β^i sphere can be obtained by the quadratic predicting curve fitted by the three previous *MPP*s $\mathbf{u}_{MPP}^{i-3}$, $\mathbf{u}_{MPP}^{i-2}$ and $\mathbf{u}_{MPP}^{i-1}$. The reason to use the quadratic approximation is that it can give us a good prediction of the next *MPP* with less computation than the higher order polynomials.

It is found that we have much better chance to find the correct *MPP* on a smaller β-sphere with the aid of the *MPP* search strategy in section 3.2.1. Our proposed method allows us to locate the *MPP*s for small β-spheres and then use them to predict the initial search point for the *MPP* on the next β-sphere with increased radius. This helps us to locate the right *MPP*s for larger β-spheres. The radius of the β-spheres are gradually increased and the trace of the *MPPs* is often referred to the *MPP* locus (the dotted curve in Fig. 5). The procedure of selecting the initial search point (Section 3.2.2) and the *MPP* search algorithm (Section 3.2.1) are integrated, which leads to an efficient and robust *MPP* search procedure.

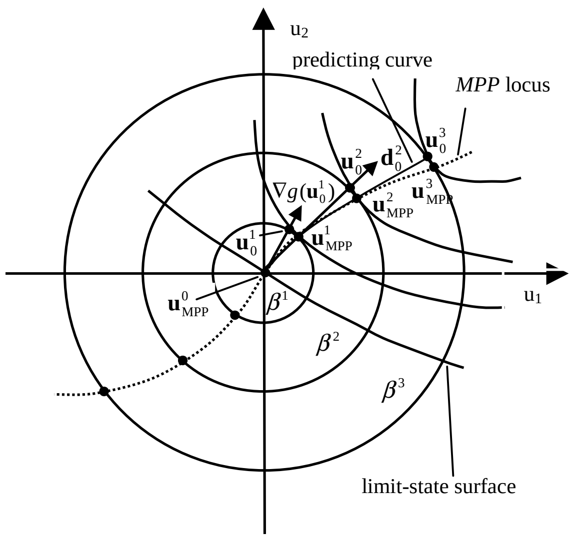


Figure 5. Tracing *MPP* Locus

4. EXAMPLES

Two examples are used to demonstrate the effectiveness of the proposed method. Example 1 is a mathematical example and example 2 is a real engineering problem. The *CDF*s and *PDF*s of the system output generated by the proposed method are compared with those obtained from the *MCS* in which very large number of simulations is considered.

4.1 Example 1 – A Mathematical Example

Two design variables $\mathbf{x} = \{x_1, x_2\}$ are considered for this example. x_1 is normally distributed with mean as 100 and standard deviation as 20; x_2 is a lognormal random variable with mean as 15 and standard deviation as 14. The system function is

$$g(\mathbf{x}) = (x_1^3 x_2 - 2x_1 x_2 - x_1^2 - 2x_2 e^{-x_1})/10000. \qquad (18)$$

CDF values obtained by the *MPPUA* and the *MCS* are compared in Table 1. The number of simulations of *MCS* is 10^7. We notice that the two sets of *CDF* values are very close to each other. Figs. 6 provides graphical comparisons of these two *CDF* curves. From Figure 6, we note that the two *CDF* curves obtained by the *MPPUA* and *MCS* almost overlap with each other. From Fig. 7, it is observed that the system output (from *MCS*) is not symmetrically distributed and this character is well captured by *PDF* curve generated using the proposed method. The two *PDF* curves are also close except there is a slight difference at the pick of the curves.

Table 1. CDF results of example 1

y	*CDF* by *MPPUA*	*CDF* by *MCS*	β
18841.07	0.9987	0.9988	3
10842.49	0.9918	0.9925	2.4
6197.639	0.9641	0.9664	1.8
3513.999	0.8849	0.8912	1.2
1973.093	0.7257	0.7374	0.6
1095.363	0.5	0.5155	0
598.3986	0.2743	0.2888	0.6
319.4687	0.1151	0.1244	1.2
165.1907	0.0359	0.0402	1.8
81.2578	0.0082	0.0096	2.4
36.58	0.0013	0.0016	3

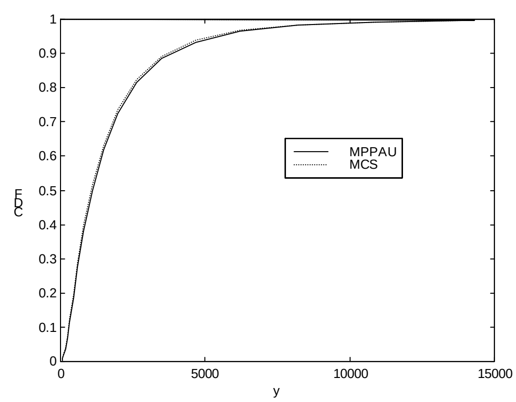


Figure 6. Example 1 – *CDF*

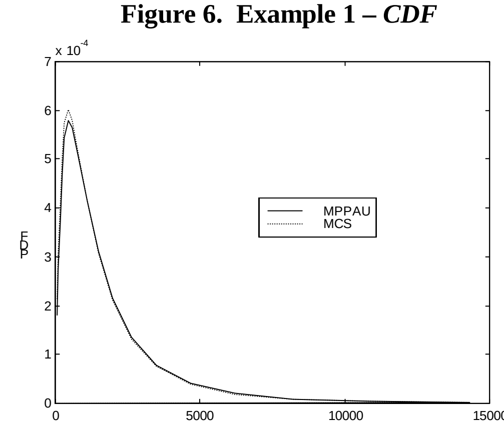


Figure 7. Example 1 – *PDF*

4.2 Example 2 – Pratt & Whitney (PW) Engine Design

The PW engine design problem is used in this study to illustrate the applicability of the proposed *MPPUA* for problems that are in a complex domain and taking a lot of computational resources. The problem statement of the PW engine design problem is provided in (Varadarajan, et al., 2000). A total of eight continuous design variables are considered. They are the Fan Pressure Ratio (FPR), the Exhaust Jet Velocity Ratio (VJR), the Turbine Inlet Temperature or the Combustor Exit temperature (CET), the High Compressor Pressure Ratio (HPCPR), the Low Compressor Pressure Ratio (LPCPR), High Turbine Compressor Efficiency (EHPC), High Turbine Efficiency (EHPT), and Low Turbine Efficiency (ELPT). The distributions of these design variables are listed in Table 2.

Table 2 Distributions of design variables in engine design

	Mean	Standard Deviation	Distribution type
FPR	0.8	0.1	Lognormal
VJR	2600	300	Lognormal
CET	15	1.5	Lognormal
HPCPR	3	0.3	Lognormal
LPCPR	1.45	0.1	Weibull
EHPC	0.89	0.08	Weibull
EHPT	0.93	0.01	Weibull
ELPT	0.805	0.01	Weibull

The system outputs are Overall Pressure Ratio (OPR), Fan Diameter (FANDIA), and HPTPR (High Turbine Pressure Ratio), all of which are nonlinear and complicated functions evaluated by a thermal analysis program called SOAPP (Varadarajan, et al., 2000).

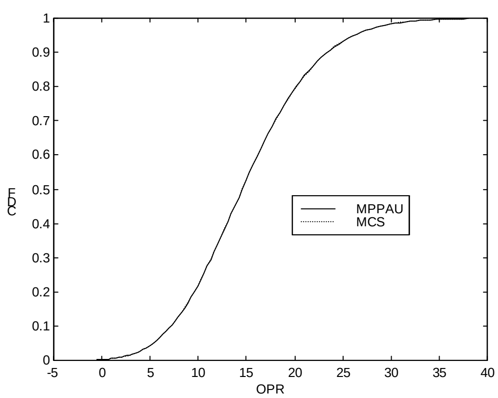


Figure 8. Example 2 – *CDF* of OPR

Figs. 8 and 9 indicate that the *MPPUA* method generates excellent estimations of *CDF* and *PDF* for OPR when the results from *MCS* are taken as a reference (simulation number is 10^7). The *MPPUA* method also generates good estimations of *CDF* and *PDF* for FANDIA even though there exist some slight differences (see Figs. 10 and 11). However for HPTPR, the estimated *CDF* and *PDF* curves are slightly shifted to the right of those obtained from *MCS* (see Figs. 12 and 13). This indicates that the model function HPTPR may have large curvatures at *MPP*s.

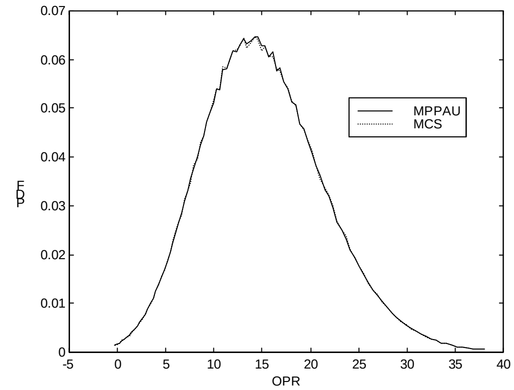


Figure 9. Example 2 – *PDF* of OPR

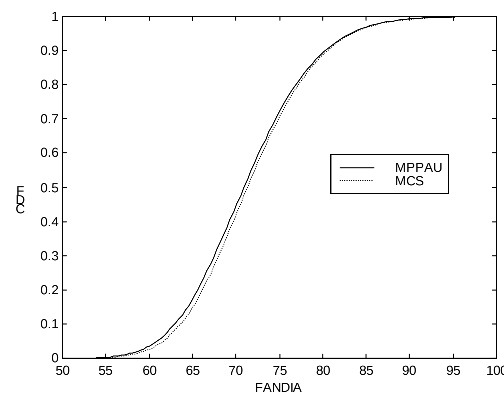


Figure 10. Example 2 – *CDF* of FANDIA

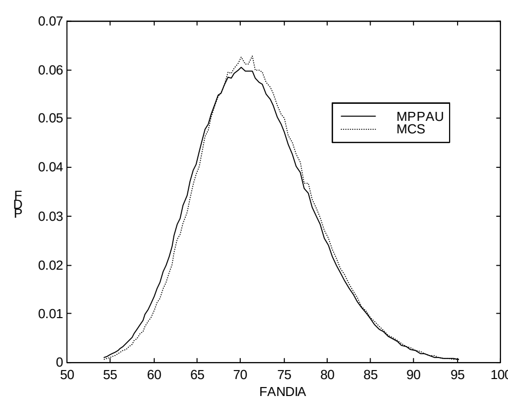


Figure 11. Example 2 – *PDF* of FANDIA

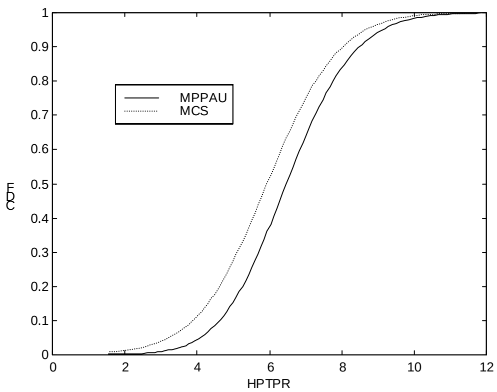


Figure 12. Example 2 – *CDF* of HPTPR

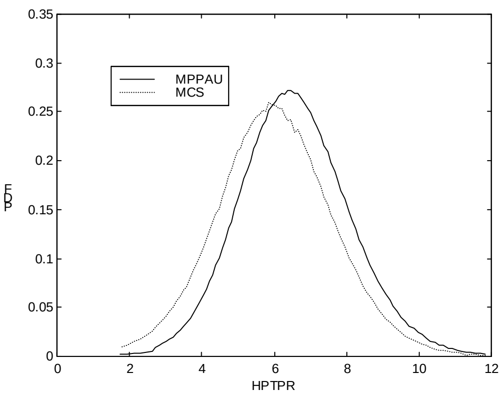


Figure 13. Example 2 – *PDF* of HPTPR

The total number of the model function evaluations is 320~500 for OPR, FANDIA and HPTPR . If the same number is taken for *MCS*, the results from *MCS* will be much less accurate than the ones from the proposed approach, especially in the tails of the *PDF*.

4.3 Discussions

Based on the two examples, we note that the *MPPUA* method provides us very good estimations of the probabilistic distribution of a system output. We comment on its performance in the following several categories:

1) Efficiency: The efficiency of the proposed method depends on the speed of *MPP* search. Only two or three iterations are needed to locate the *MPP* at each β for the above two examples. This means that after the initial point is identified, only two or three points on the β-sphere need to be updated before the *MPP* is located. Our proposed search strategy (section 3.2.1) and the method of selecting the initial search point along the *MPP* locus (section 3.2.2) have significantly enhanced the performance of utilizing the *MPP* concept.

2) Robustness: Tracing the *MPP* locus is very helpful for the convergence of *MPP* search when dealing with different types of system behavior because it always gives us good initial search points to start with. As discussed in section 3.2.2, we start *MPP* search on the smallest β-sphere. It is easy to find a correct *MPP* on a small β-sphere with the aid of the *MPP* search strategy. The initial point for the *MPP* search on the larger β-sphere is predicted using the *MPPs* for smaller β-spheres. Good starting points help to converge to the right *MPP* points in an efficient manner.

3) Simplicity: The *MPPUA* method is easy to be computerized.

In spite of the advantages listed, we should keep in mind that our proposed method is only an approximation method for predicting the probabilistic distributions in a certain situations. Its accuracy depends on many issues such as the linearity of the transformed system function at the *MPP*, the normalization of the non-normal random variables, the stopping criteria of the *MPP* search, etc. The linearity is the key factor that affects the accuracy. Highly nonlinear model function (large magnitude of the principal curvatures at *MPP* in **u** space) may result in significant errors of *CDF* estimation. To improve the accuracy under this situation, the second-order approximation to the probability $P\{g(\mathbf{x}) < y_i\}$ can be used to fit the model function at the *MPP*. Usually a paraboloid or a sphere function (Fiessler et al., 1979; Hohenbichler, et al., 1987) is used. The simplest of these methods, based on a paraboloid fitting, is the asymptotic formulation (Breitung, 1984):

$$P[g(\mathbf{x}) \le y_i] = \Phi(-\beta^i)\prod_{j=1}^{n-1}(1+\beta^i\rho_j)^{-1/2} . \qquad (19)$$

where ρ_j denote the main curvatures of the model function or limit-state function at the *MPP*. When there exist more than one local minimal distance points (multiple *MPP*s), an approximation of the solution can be obtained by fitting planes or paraboloid surfaces at all these points (Kiureghian, 1987). Obviously, the improvement of accuracy will involve the evaluations of second derivatives of a function. To avoid this, a *MPP* based importance sampling method can be used. This method is implemented by sampling around the *MPP* rather than over the whole random space. Therefore the number of simulation is much less than that of the general *MCS*. This method is fully documented in Du and Chen, 1999b.

While the accuracy of the *MPPUA* method mainly depends on the features of a system model, its efficiency relies on the problem dimension (the number of random variables). Since the *MPPUA* method is a gradient based method and generally the gradient is evaluated by numerical methods, the number of function evaluations will be approximately equal to the number of random design variables times the number of derivative updates. Therefore, when the number of random variables is very large (for example, bigger than 50), the advantage of the *MPPDG* method over simulation methods may diminish. In this situation, *MCS* may be more efficient to use than the *MPPUA* method. To deal with large-scale problems, one solution is to use the design of experiments techniques to screen out unimportant variables (Box, et al., 1978) and then to implement the *MPPUA* method in the reduced design space.

5. CLOSURE

Adopting probabilistic model in engineering design is very important for designers to make intelligent and reliable decisions under uncertainty. Uncertainty analysis is required to propagate the effect of uncertainty on a system output. The research of this paper is focused on how to capture the complete probabilistic distribution of a system output with the existence of uncertainty in system input. To accomplish this goal, a *MPP* based uncertainty analysis method is proposed. The method utilizes the concept of *MPP* to generate the cumulative distribution function (*CDF*) of a system output with any types of continuous distributions for input random variables by evaluating probability estimate at the a serial of limit states. The probability density function (*PDF*) can then be estimated by the differential of *CDF*. A novel *MPP* search algorithm is presented that employs a set of searching strategies, including adjusting search directions based on derivatives, tracing the *MPP* locus, and predicting the initial point for *MPP* search. With the capability of generating the probabilistic distribution of a system output, the proposed method can serve as a useful tool in decision making under uncertainty. In addition to utility function optimization, it can also be used in many forms of probabilistic design, such as robust design, stochastic optimization, and reliability based design. The proposed method can be used to assist the evaluation of mean and variance (both are major concerns in robust design), the probability of success (such as the limit-state evaluation in reliability analysis), and the feasibility robustness (if the system output is considered as the constraint). Our future study will be towards integrating the proposed uncertainty analysis method with a decision making framework.

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REFERENCES

Ang, A.H.-S. and Tang, W.H., 1984, *Probabilistic Concepts in Engineering Planning and Decision*, Vol. II, John Wiley and Sons, New York.

Ayyub, B.M. and Chao R.U., 1997, “Uncertainty Modeling in Civil Engineering with Structural and Reliability Applications,” *Uncertainty Modeling and Analysis in Civil Engineering*, B.M., CRC Press, pp. 1-8.

Box, G. E. P., Hunter, W. G. and Hunter, J. S., 1978, *Statistics for Experiments*, John Wiley & Sons, New York.

Breitung, K., 1984, “Asymptotic Approximation for multinormal Integrals,” *ASCE J. Eng. Mech*., Vol. 110, pp. 357-366.

Brown, S.A. and Sepulveda, A.E., 1997, “Approximation of System Reliability Using a Shooting Monte Carlo Approach,” *AIAA journal*, Vol. 35, No. 6, pp. 1064-1071.

Cagan, J., and Williams, B.C., 1993, “First-Order Necessary Conditions for Robust Optimality,” *Advances in Design Automation*, ASME DE-Vol. 65-1, pp.539-549.

Chen, W., Allen, J.K., Mistree, F., and Tsui, K.-L., 1996, “A Procedure for Robust Design: Minimizing Variations Caused by Noise Factors and Control Factors,” *ASME Journal of Mechanical Design*, Vol. 118, pp.478-485.

DeLaurentis D. and Marvris D., 2000, “Uncertainty Modeling and Management in Multidisciplinary Analysis and Synthesis,” *The 38th AIAA Aerospace Sciences Meeting*, Paper No. AIAA 2000-0422, 10-13 January 200, Reno, NV.

De Neufville S., 1990, *Applied Systems Analysis, McGraw-Hill*, Inc., Singapore.

Ditlevsen, O., Olesen, R. and Mohr, G., 1987, “Solution of A Class of Load Combination Problems by Directional Simulation,” *Structural Safety*, No. 4, pp. 95-109.

Du, X. and Chen, W., 1999a, “Propagation and Management of Uncertainties in Simulation-Based Collaborative Systems Design,” *The 3rd World Congress of structural and Multidisciplinary Optimization*, May 17-21, 1999, Buffalo, New York, pp.390-392.

Du, X. and Chen, W., 1999b, “Towards a Better Understanding of Modeling Feasibility Robustness in Engineering Design,” *1999 ASME Design Technical Conference*, Paper No. DAC-8565, Las Vegas, Nevada, Sept. 1999.

Du, X. and Chen, W., 2000, “An Efficient Approach To Probabilistic Uncertainty Analysis In Simulation-Based Multidisciplinary Design,” *The 38th AIAA Aerospace Sciences Meeting*, Paper No. AIAA 2000-0423, 10-13 January 200, Reno, NV.

Du, X. Wang Y. and Chen, W., 2000, "Method for Robust Multidisciplinary Design," the 41st AIAA /ASME /ASCE/ AHS/ ASC Structures, Structural Dynamics, and Materials Conference, AIAA, Washington D.C.

Eggert, R.J., 1991, “Quantifying Design Feasibility Using Probabilistic Feasibility Analysis,” *1991 ASME Advances in Design Automation*, Paper No. DE-Vol. 32-1, pp. 235-240.

Fiessler, B., Neuman, H-J. and Rackwitz, R., 1979, “Quadratic Limit States in Structural Reliability,” *Journal of Engineering Mechanics*, ASCE, Vol. 105, No. 4, pp. 661-676.

Gu, X, Renaud, J. E., and Batill, S. M., 1998, “An Investigation of Multidisciplinary Design Subject to Uncertainties,” *7th AIAA/USAF/NASA/ISSMO Multidisciplinary Analysis & Optimization Symposium*, St. Louise, Missouri, 1998, pp. 309-319.

Hasofer, A.M. and Lind, N.C., 1974, “Exact and Invariant Second-Moment Code Format,” *Journal of the Engineering Mechanics Division*, ASCE, Vol. 100, pp. 111-121.

Hazelrigg, G.A., 1999, “An Axiomatic Framework for Engineering Framework for Engineering Design,” *ASME Journal of Mechanical Design*, Vol. 121, pp.342-347.

Hazelrigg, G.A., 1998, "A Framework for Decision-Based Engineering Design," *ASME Journal of Mechanical Design*, Vol. 120, pp.653-658.

Hazelrigg, G.A., 1996, *Systems Engineering: An Approach to Information-Based Design*, Prentice Hall, New Jersey.

Hohenbichler, M., Gollwitzer, S., Kruse, W. and Rackwitz, R., 1987, "New Light on First – and Second- Order Reliability Methods," *Structural Safety*, No. 4, pp. 267-284.

Isukapalli, S.S, and Georgopoulos, P.G., 1998, "Stochastic Response Surface Methods (SRSMs) for Uncertainty Propagation: Application to Environmental and Biological Systems," *Risk Analysis*, Vol. 18, pp. 351-363.

Isukapalli, S.S, 1999, *Uncertainty Analysis of Transport-Transformation Models*, Ph.D. Thesis, January, 1999, Rutgers, The State University of New Jersey

Iyer, H.V. and Krishnamurty, S., 1998, "A Preference-Based Robust Design Metric", *1998 ASME Design Technical Conference*, Paper No. DAC5625, Atlanta, GA.

Kahn, H., 1956, "Use of Different Monte Carlo Sampling Techniques," *Proc. Symp*. o*n Monte Carlo Methods*, Editor: Meyer, H. A., John Wiley & Sons, New York, pp. 149-190.

Kiureghian, A.D., Lin, H.-Z. and Hwang, S.-J., 1987, "Second-order Reliability Approximations," *Journal Engineering Mechanics*, Vol. 113, No. 8, pp. 1208-1225.

Koch, P.N., Simpson, T.W., Allen, J.K., and Mistree F., 1999, "Statistical Approximations for Multidisciplinary Design Optimization: The Problem of Size," *Journal of Aircraft*, Vol. 36, No. 1, pp. 275-286.

Kowal, M. and Mahadevan S., 1998, "Uncertainty-Based Multidisciplinary Design Optimization," 7TH AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary *Analysis and Optimization*, St. Lois, MO.

Laskey, K. B., 1996, "Model Uncertainty: Theory and Practical Implications," *IEEE Transactions on System, Man, and Cybernetics – Part A: System and Human*, Vol. 26, No. 3, pp. 340-348.

Manners W., 1990, "Classification and Analysis of Uncertainty in Structural System," *Proceedings of the 3rd IFIP WG 7.5 Conference on Reliability and Optimization of Structural Systems, Berkeley*, California, March, pp.251-260.

Mavris, D.V, Bandte, O. and DeLaurentis, D.A., 1999, "Robust Design Simulation: A Probabilistic Approach to Multidisciplinary Design," *Journal of Aircraft*, Vol. 36, No. 1, pp. 298-397.

Melchers, R.E., 1999, *Structural Reliability Analysis and Prediction*, John Wiler & Sons, Chichester, England.

Mitteau, J.-C., 1999, "Error Evaluations for the Computation of Failure Probability in Static Structural Reliability Problems," *Probabilistic Engineering Mechanics*, Vol. 14, No. 1/2, pp. 119-135.

Niederreiter H., 1992, *Random Number Generation and Quasi-Monte Carlo Methods*, Society for Industrial and Applied Mathematics, Philadelphia.

Parkinson, A., Sorensen, C., and Pourhassan, N., 1993, "A General Approach for Robust Optimal Design," *Transactions of the ASME*, Vol. 115, pp.74-80.

Papoulis A., 1991, *Probability, Random Variables, and Stochastic Process*, WCB/McGraw-Hill, Boston.

Pinto, J.C., 1998, "On the Costs of Parameter Uncertainties. Effects of Parameter uncertainties During Optimization and Design of Experiments," *Chemical Engineering Science*, Vol. 53, No. 11, pp. 2029-2040.

Rosenblatt, M., 1952, "Remarks on a Multivariate Transformation," *Annal of Mathematical Statistic*, Vol. 23, pp. 470-472.

Rubinstein, R.Y., 1981, *Simulation and Monte Carlo Method*, John Wiley & Sons, New York.

Sues, R. H., Oakley, D. R. and Rhodes, G. S., 1995, "Multidisciplinary Stochastic Optimization," *Proceedings of the 10th Conference on Engineering Mechanics*, Part 2, Vol. 2, Boulder, CO., May 1995.

Tichy, M., 1993, *Applied Methods of Structural Reliability*, Kluwer Academic Publishers Norwell.

Tvedt L., 1990, "Distribution of Quadratic Forms in Normal Space - Application to Structural Reliability," *ASCE Journal of Engineering Mechanics*, Vol. 116, No. 6, pp. 1183-1197.

Varadarajan S., Chen W. and Pelka, C., 2000, "The Robust Concept Exploration Method with Enhanced Model Approximation Capabilities," *Journal of Engineering Optimization*, Vol. 32, No. 3.

Walker, J. R, 1986, "Practical Application Of Variance Reduction Techniques In Probabilistic Assessments," *the Second International Conference on Radioactive Waste Management*. Winnipeg, Manit, Canada, pp. 517-521.

Wang C., 1999, *Parametric Uncertainty Analysis for Complex Engineering Systems*, Ph.D. Thesis, June, MIT.

Wu, Y.-T., Millwater, H.R., and Cruse T. A., 1990, "An Advance Probabilistic Analysis Method for Implicit Performance Function," *AIAA Journal*, Vol. 28, pp. 1663-1669.

Wu, Y.-T., 1998, "Methods for Efficient Probabilistic Analysis of System with Large Numbers of Random Variables," 7Th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis Optimization, September 15-18, 1998, St. Louis, MO, Paper No. AIAA 98-4908.