QUALITY UTILITY – A COMPROMISE PROGRAMMING APPROACH TO ROBUST DESIGN

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ABSTRACT

In robust design, associated with each quality characteristic, the design objective often involves multiple aspects such as "bringing the mean of performance on target" and "minimizing the variations". Current ways of handling these multiple aspects using either the Taguchi's signal-to-noise ratio or the weighted-sum method are not adequate. In this paper, we solve bi-objective robust design problems from a utility perspective by following upon the recent developments on relating utility function optimization to a Compromise Programming (CP) method. A robust design procedure is developed to allow a designer to express his/her preference structure of multiple aspects of robust design. The CP approach, i.e., the Tchebycheff method, is then used to determine the robust design solution which is guaranteed to belong to the set of efficient solutions (Pareto points). The quality utility at the candidate solution is represented by means of a quadratic function in a certain sense equivalent to the weighted Tchebycheff metric. The obtained utility function can be used to explore the set of efficient solutions in a neighborhood of the candidate solution. The iterative nature of our proposed procedure will assist decision making in quality engineering and the applications of robust design.

Keywords: Robust Design, Multiobjective Optimization, Compromise Programming, Utility Function, Decision Analysis

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NOMENCLATURE

$CP(\infty, w)$	Weighted Tchebycheff Approach
$f(x)$	Objective Function
$F(x)$	Vector of Objective Functions
$\bar{F}$	Candidate Efficient Solution
$g(x)$	Constraint Function
k_j	Penalty Factors
K	Loss Function Coefficient
L_1	Manhattan Metric
L_2	Euclidean Metric
L_∞	Tchebycheff Metric
L_p	L_p-metric
S/N	Signal/Noise
w	Weights
WS	Weighted-sum
WSP(w)	Weighted-sum Problem
x	Vector of Design Variables
X	Design Space
x_L	Lower Bound for Design Variables
x_U	Upper Bound for Design Variables
x^0	Pareto Solution for Design Variables
x^*	Optimal Solution for Design Variables
$\bar{x}$	Candidate Solution in Design Space
Y	Random Variable; Objective Space
Δx	Deviations of the Design Variables
β^*	Optimal Solution of β-problem
η	Quality Characteristic of S/N
u^*	Utopia Point in CP
μ_f	Mean of the Objective Function f(x)
μ_f^*	Optimal Value of the Mean
σ_f	Standard Deviation of the Objective Function f(x)
σ_f^*	Optimal Value of the Standard Deviation

1. INTRODUCTION

In recent years, the Taguchi robust design method has been widely used to design quality into products and processes (Phadke, 1989). Using this method, the quality of a product is improved by minimizing the effect of the causes of variation without eliminating the causes (Taguchi, 1993). While the majority of the early applications of robust design consider manufacturing as the cause for performance variations, recent developments in design methodology have produced approaches that utilize the same concept to improve the robustness of design decisions with respect to the variations associated with the design process (Chang et al., 1994; Chen et al., 1996b).

Although Taguchi's robust design principle has been widely accepted, the methods Taguchi offers for robust design have received much criticism, in particular the two-part orthogonal array for experimental design and the signal-to-noise-ratio (S/N ratio) used as the robust optimization criterion (Box, 1988; Nair, 1992). In the engineering design community, researchers are working on developing nonlinear programming methods that can be used for a variety of robust design applications (Otto and Antonsson, 1991; Parkinson et al., 1993; Yu and Ishii, 1994; and Cagan and Williams, 1993), including probabilistic optimization methods for robust design (Siddall, 1984; Eggert and Mayne, 1993). A comprehensive review of existing robust optimization methods is provided by Su and Renaud (1997), and will not be repeated here.

One issue that we find has not been adequately addressed in the previous investigations is the *multiple aspects* of the objective in robust design. It was illustrated by one of the authors (Chen et al., 1996b) that associated with each quality (performance) characteristic, the robust design objective could be generalized into *two aspects*, namely, “optimizing the mean of performance” and “minimizing the variation of performance”. A brief mathematical background that supports the above statement is provided in Section 2.1. Through our previous applications (Chen et al., 1996a and Chen et al., 1997), we observe that the performance variation is often minimized at the cost of sacrificing the best performance, and therefore the tradeoff between the aforementioned two aspects cannot be avoided. In the literature, though the multiple aspects of the objective in robust design is acknowledged (Sundaresan et al., 1993), *single* robust design objective function is often utilized. Ramakrishnan and Rao, 1991, formulate the robust design problem as a nonlinear optimization problem with Taguchi's loss function as the objective. Sundaresan et al. (1993) employ a *single* objective function that utilizes *weighting* factors for target performance and variance represented by the Sensitivity Index (SI). Bras and Mistree (1995) and Chen et al. (1996b) introduce the compromise Decision Support Problem (DSP) (Mistree et al., 1993), a goal programming approach, to model the multiple aspects of robust design objective as separate goals. We assert that the use of weighted sums of objectives is a very simplistic approach to multiobjective optimization problems. A closer look at the drawbacks of minimizing weighted sums of objectives in multicriteria optimization is provided by Das and Dennis (1997). More rigorous methods need to be considered for representing the preference structure of multiple objectives in robust design.

For modeling designer’s preference structure, one of the commonly used methods is based on the utility theory (von Neumann and Morgenstern, 1947; Keeney and Raifa, 1976; Hazelrigg, 1996; Thurston, 1991). Under the notion of utility theory, the ultimate overall worth of a design is represented by a multiattribute utility function which incorporates consideration of attributes that cannot be directly converted to a common metric. Ideally, when the preference of the multiple aspects of the objective in robust design could be captured by the multiattribute utility analysis, robust design could be solved as a single objective optimization problem. However, one difficulty associated with using the utility function approach is that, in practice, it is often impossible to obtain a reliable mathematical representation of the decision-maker's actual utility function. In the literature, approaches that take different paradigms for solving multicriteria optimization problems are proposed. For instance, Messac (1996) develops the method of physical programming which eliminates the need for weight setting or utility function building in multicriteria optimization. In this work, we propose to use Compromise Programming (CP) (Yu, 1973 and Zeleny, 1973) to address the multiple aspects of robust design.

CP is one of the approaches that take a paradigm different from the utility theory. The basic idea in CP is the identification of an ideal solution as a point where each attribute under consideration achieves its optimum value and seek a solution that is as close as possible to the ideal point (Zeleny's axiom of choice). Though the weights representing relative importance are used as the preference structure when applying CP, it has been mathematically proven that CP is superior to the weighted-sum (WS) method in locating the efficient solutions, or the so called Pareto points (Steuer, 1986). However, there are few applications of CP to mechanical engineering design problems. Miura and Chargin (1996) develop a variation of CP and apply it to optimal structural design. Athan and Papalambros (1996) do not refer to CP but propose to minimize the sum of the exponentially weighted objective functions and illustrate their approach also on some structural design problems.

Though utility theory and CP are considered very different paradigms and methodologies to measure preferences as well as to determine decision maker's optima on the efficient frontier, researchers have illustrated a linkage between the two approaches (Ballestero and Romero, 1991). One of the authors established a relationship between a CP approach and a quadratic weighted-sums scalarization of multiobjective problems (Tind and Wiecek, 1997). In this paper, we apply CP, specifically the Tchebycheff method, to multiobjective robust design problems from a utility perspective by following upon the recent developments. An interactive robust design procedure is developed to support decision making in robust design applications.

2. TECHNOLOGICAL BASIS OF OUR APPROACH

2.1 Multiple Quality Aspects of Robust Design

The quality loss function is used by Taguchi as a metric for robust optimization. The relationship between quality loss and the amount of deviation from the target value is expressed by the loss functions for different types of quality characteristics (“the nominal the better”, “the smaller the better”, and “the larger the better”). For "the nominal the better” type of problem, it was proven (Tsui, 1992) that the expected quality loss can be obtained as

$$E\,[L(Y,T)] = K\,[\sigma^2 + (T-\mu)^2], \tag{2.1}$$

where K is the loss function coefficient, Y is the random variable of quality characteristic y, T is the target or desired value of the quality characteristic, σ^2 is the variance of the random variable Y, and μ is the mean of the population.

From Eqn. 2.1, it is noted that minimizing the expected quality loss can be achieved by minimizing both the variance σ^2 and the difference between the mean and the target, i.e., $T - \mu$. As illustrated in Figure 1, robust design is always concerned with aligning the peak of the bell shaped response distribution with the targeted quality (optimizing the mean performance), and making the bell shaped curve thinner (minimizing the variance σ). To address the multiple aspects of robust design, it is necessary to treat it as a multiobjective optimization problem.

Insert Figure 1 Quality Distribution in Robust Design

2.2 CP vs. WS Method

In this section we review and compare two approaches to finding its Pareto set: the WS approach and CP. We show limitations of the former and advantages of the latter.

Consider the multiobjective optimization problem formulated as

$$\begin{aligned} &\text{minimize} && F(x) \\ &\text{subject to} && x \in X \subset R^n, \end{aligned} \tag{2.2}$$

where $F(x) = [f_1(x), ..., f_m(x)]$ and $f_i(x)$, $i = 1, ..., m$, are real-valued continuous functions defined in R^n. Let X denote the design space that is formed by both the design constraints and the range of design variables x, and $Y = F(X) \subset R^m$ be the objective space. If the objective functions remain in conflict over the design space, then it is impossible to find a point at which they would assume their minimum values simultaneously, and consequently, the classical concept of a common optimal solution does not apply. In this situation, the concept of Pareto solutions is exercised. A point x^0 is called a *Pareto solution* of the problem (Eqn. 2.2) if there is no other feasible point x such that $f_i(x) \le f_i(x^0)$, $i = 1, ..., m$, with strict inequality for at least one index i. If the objective functions are in conflict, then in general, there may be infinitely many Pareto solutions. The image $F(x^0)$ of a Pareto solution x^0 in the objective space is called the *efficient solution*. As the set of all efficient solutions is always located on the boundary of Y, it is also referred to as the *efficient frontier*. Minimizing each objective function individually over the design space we obtain

$$f_i^{min} = \text{minimum } \{ f_i(x), x \in X\}, i = 1, ..., m, \tag{2.3}$$

which yields a *utopia (ideal) point* $u \in R^m$ defined as

$$u_i = f_i^{min} - \varepsilon_i, i = 1, ..., m. \tag{2.4}$$

where $\varepsilon_i \ge 0$. Due to the conflict between the objective functions, the utopia point is never in Y.

The common practice for finding Pareto solutions has been the WS method that performs the minimization of a linear combination of the objective functions. The corresponding *weighted-sum problem* (WSP(w)) is

$$\begin{aligned} &\text{minimize} && \sum_{i=1}^{m} w_i f_i(x) \\ &\text{subject to} && x \in X \subset R^n, \end{aligned} \tag{2.5}$$

where $w_i \geq 0$, $i = 1, ..., m$, and $\sum_{i=1}^{m} w_i = 1$. Scalars w_i are referred to as the weights assigned to the objective f_i, $i = 1, ..., m$, and determine the importance of each objective. *An optimal solution of the WSP(w) for any positive weights is always a Pareto solution of the original problem (2.3). However, not every Pareto solution can be found by solving the WSP(w),* that is there may not exist a weight w such that a given Pareto point may be found by solving the WSP(w). Figure 2 shows the efficient set (frontier) of a bi-objective problem in the objective space. The solutions found by solving the WSP(w) can be geometrically identified as the points of contact between the curve and the line supporting the curve and perpendicular to the vector w**.** This figure shows that WSP (w) may fail to generate the efficient solutions located on the arc between points A and B, since for some vector $w \geq 0$, it could achieve a better (smaller) weighted sum value by supporting the Pareto curve outside of the arc rather than at any point along that arc.

Insert Figure 2 Generating Efficient Solutions by the WS method

On the other hand, the WS method works very well for convex problems, i.e., such whose objective functions are convex and design spaces are convex sets. Due to Geoffrion (1968), for every Pareto point of a convex problem there exists a weight $w \geq 0$ such that this Pareto point is an optimal solution of the WSP(w). The disadvantage of the WS method while dealing with nonconvex problems has been recognized already in the early seventies by several researchers. Moreover, it is found that for convex multiobjective optimization problems, an even spread of weights w does not produce an even spread of points in the efficient set. Motivated by the obvious need of looking for a more powerful approach, Zeleny (1973), Yu and Leitmann (1974), Bowman (1976), and others developed Compromise Programming (CP), an approach based on a procedure that finds an efficient point closest to the utopia point.

Since to measure the distance between an efficient point and the utopia point one may use different metrics (mathematical measures of distance between points, see Steuer, 1986), the general *compromise programming problem* is formulated as:

$$\begin{aligned} &\text{minimize} && \| f(x) - u \| \\ &\text{subject to} && x \in X, \end{aligned} \tag{2.6}$$

where $\| \cdot \|$ denotes the metric of choice. For a *weighted L_p-metric*, the distance between two points r, s in R^m is given by:

$$\| r - s \|_p^w = \left(\sum_{i=1}^{m} (w_i |r_i - s_i|)^p\right)^{1/p}, \tag{2.7}$$

where $p = \{1, 2, ...\} \cup \{ \infty \}$ defines the metric and $w_i \geq 0$, $i = 1, ..., m$, are again the weights representing the relative weight or worth of one objective against another. In particular, for $p = 1$, the definition yields the so called Manhattan metric, L_2 is the Euclidean metric, and L_∞ is the Tchebycheff metric. As the structure of the CP problem depends upon the choice of the metric, we use the notation CP(p,w). Observe that since for every $x \in X$, $u_i \leq f_i(x)$, $i =1, ..., m$, the absolute value in the definition of the metrics can be dropped, consequently, the CP(1, w) is *equivalent to* the WS formulation. Let $2 \leq p < \infty$. Then the objective function of the CP(p, w) is nonlinear and not easy to handle. However, for $p = \infty$, we have the CP(∞, w) as a min-max problem:

$$\min_{x \in X} \max_{i=1, ..., m} \{ w_i(f_i(x) - u_i) \}, \tag{2.8}$$

which is equivalent to the following *b-problem*

$$\begin{aligned} &\text{minimize} && \beta \\ &\text{subject to} && w_i (f_i(x) - u_i) \leq \beta, \; i = 1, ..., m \\ &&& x \in X, \end{aligned} \tag{2.9}$$

where β is a new, always positive variable.

The CP(∞, w), referred to as *the weighted Tchebycheff approach,* turns out to be very useful in generating Pareto solutions. Bowman (1976) shows that for every Pareto solution there exists a positive vector of weights so that the corresponding CP(∞, w) is solved by this Pareto point. Figure 3 shows the same efficient frontier that is depicted in Figure 2. For the given utopia point u and the vector w, the solutions of CP(∝, w) can be geometrically identified as the points of contact between the efficient frontier and the corresponding level curve of the weighted Tchebycheff metric. We observe that keeping the utopia point the same but changing the weights, one may reach all the efficient points located on the arc between points A and B. In this paper, the CP(∝, w) is used to solve a multiobjective robust design problem (MORD) which guarantees that all efficient solutions of this problem can be generated.

Insert Figure 3 Generating Efficient Solutions by the Weighted-Tchebycheff Method

2.3 Relating Utility Function Optimization to CP

Tind and Wiecek (1997) show that under certain conditions, the Pareto solution found by means of the Tchebycheff approach for a given utopia point and a given vector of weights can also be generated through the minimization of a quadratic function of the original objective functions. This quadratic function provides a decision maker with new information regarding how to choose the most preferred Pareto solution. Although the decision maker's utility is typically maximized, in this approach it is minimized due to the fact that the original bi-objective problem involves minimization. The main role of this quadratic utility function is to identify solutions of equal utility to the decision maker in a neighborhood of the efficient solution found by the Tchebycheff approach. More specifically, in our application, we use this quadratic function as an approximation of the efficient frontier in the neighborhood of the candidate solution to explore the alternative robust design solutions.

Due to the nature of the robust design problem, we focus on the bi-objective case of problem (2.3). We assume that X is a compact set and the objective functions are twice continuously differentiable. Let $w_i > 0$, i = 1, 2, and (x^*, β^*) be an optimal solution of the β-problem given by Eqn. (2.9), so that the inequality constraints are binding at optimality

$$w_i(f_i(x^*) - u_i) = \beta^*, \; i = 1, 2. \tag{2.10}$$

We also assume that the second order sufficiency conditions for optimality hold at (x^*, β^*) (see Bazaraa et al. (1993)). The point $x^* \in X$ is Pareto and its image $F(x^*) = [f_1(x^*), f_2(x^*)]$ in the objective space is efficient and has components $u_i^* - (\beta^*/w_i)$, i = 1, 2. According to the investigations by Tind and Wiecek (1997), the same Pareto point is also an optimal solution of the following quadratic-weighted sum problem (QWSP(Q, p)) :

$$\begin{array}{ll} \text{minmize} & F(x)^T Q F(x) + p^T F(x) + c \\ \text{subject to} & x \in X, \end{array} \tag{2.11}$$

where Q is a symmetric 2 x 2 matrix of the form

$$Q \;=\; (1/2)\,\alpha \begin{bmatrix} w_1^2 & -w_1 w_2 \\ -w_1 w_2 & w_2^2 \end{bmatrix}, \tag{2.12}$$

p is a vector, $p \in R^2$:

$$p = \begin{bmatrix} -\alpha w_1^2 u_1^* + \alpha w_1 w_2 u_2^* + w_1 y_1 \\ -\alpha w_2^2 u_2^* + \alpha w_1 w_2 u_1^* + w_2 y_2 \end{bmatrix}, \tag{2.13}$$

and

$$c = (u^*)^T Q\, u^* + w_1 y_1\, u_1^* + w_2 y_2\, u_2^*. \tag{2.14}$$

The matrix Q, the vector p, and the scalar c involve new parameters: α and y; α is a positive scalar sufficiently large and $y \in R^2$ is entirely determined by the weights:

$$(y_1, y_2) = \left(\frac{w_2^2}{w_1^2 + w_2^2}, \frac{w_1^2}{w_1^2 + w_2^2}\right). \tag{2.15}$$

It can be shown that the matrix Q is positive semi-definite so that the quadratic problem involves the minimization of a convex quadratic function of the objective functions over the image of the design space in the objective space. As a result, the decision maker's utility represented by the weighted Tchebycheff metric achieves a new representation by means of this convex function**.** The weighted-Tchebycheff problem is easy to solve while the quadratic weighted-sum problem is computationally complex. As the two problems generate the same Pareto point, solving the former yields this solution while just formulating (not solving) the latter provides the quadratic function. This utility function is more attractive to the decision maker than the function represented by the level curves of the weighted Tchebycheff metric. In a neighborhood of the efficient point found, the quadratic function provides the decision maker with finite trade-offs and specifies the points of equal utility.

3. A ROBUST DESIGN PROCEDURE UTILIZING COMPROMISE PROGRAMMING AND THE UTILITY FUNCTION

Based on the theoretical foundations of the CP approach and the method of relating utility function optimization to it, a robust design procedure is proposed in this paper to address the multiple aspects of robust design, see Figure 4.

Insert Figure 4 A Robust Design Procedure for Addressing Multiple Aspects of Robust Design

To start, the first step is to transform a conventional optimization problem into a robust design formulation. We state an engineering design optimization problem using the following conventional optimization model:

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{stubject to} && g_j(x) \le 0, \qquad j = 1,\ 2,\ ...,\ J \\ &&& x_L \le x \le x_U, \end{aligned} \qquad (3.1)$$

where x, x_L and x_U are vectors of design variables, their lower bounds and upper bounds, respectively; f(x) stands for the objective function and $g_j(x)$ is the j-th constraint function.

In robust design, both the robustness of the objective function and the robustness of the constraint functions are considered. The robust design is stated as a bi-objective robust design (BORD) problem as the following:

$$
\begin{aligned}
&\text{Minimize} && [\ \mu_f, \sigma_f\ ] \\
&\text{subject to} && g_j(x) + k_j \sum_{i=1}^{n} \left| \frac{\partial g_j}{\partial x_i} \right| \Delta x_i \le 0,\ \ j=1, 2, ..., J \qquad (3.2) \\
& && x_L + \Delta x \le x \le x_U - \Delta x,
\end{aligned}
$$

where μ_f and σ_f are the mean and standard deviation of the objective function f(x), respectively. Their values can be obtained through statistical analyses based on simulations or the first-order Taylor expansion if the design deviations of x_i are small. In Eqn. 3.2, to study the variation of constraints, we use the worst case scenario. It is assumed that all variations of system performance may occur simultaneously in the worst possible combination of design variables. The original constraints are modified by adding the penalty term to each of them, where k_j (j=1, …,J) are penalty factors to be determined by the designer. The bounds of design variables are also modified to ensure feasibility under deviations.

Once the problem is transformed into a BORD problem, the next step is to seek the ideal solution (utopia point) by optimizing μ_f and σ_f individually using the model in Eqn. 3.2. While μ_f is optimized using either "the smaller the better", "the nominal the best", and "the larger the better", it is always desired to minimize σ_f. We denote $[\mu_f^*, \sigma_f^*]$ as the obtained utopia point.

With the knowledge of the ideal solution of the robust design problem, the designer needs to specify a preference structure by assigning weights w_1 and w_2 representing the relative importance of the two objectives. Using CP, namely, the weighted-Tchebycheff method, the BORD problem can be formulated using the following form:

$$\text{minimize} \quad b$$

$$\text{subject to} \quad w_1\left(\frac{\mu_f}{\mu_f^*} - 1.0 + \varepsilon_1\right) \le \beta$$

$$w_2\left(\frac{\sigma_f}{\sigma_f^*} - 1.0 + \varepsilon_2\right) \le \beta$$

$$g_j(x) + k_j \sum_{i=1}^{n} \left|\frac{\partial g_j}{\partial x_i}\right| \Delta x_i \le 0, \quad j=1, 2, ..., J \qquad (3.3)$$

$$x_L + \Delta x \le x \le x_U - \Delta x.$$

Note that the two objective functions of interest are now (μ_f / μ_f^*) and (σ_f / σ_f^*). As discussed in Section 2.2, the solution obtained from the above model is guaranteed to belong to the Pareto solutions of the BORD problem. At this point, the designer is asked whether s/he is satisfied with the candidate solution, denoted as $\bar{x}$. If the answer is yes, the robust design procedure stops. Otherwise, the designer will be given the option of either exploring the set of efficient solutions in a neighborhood of the $\bar{x}$ or modifying the preference structure to solve again the BORD using CP. When the designer is interested in exploring the efficient solutions in a neighborhood of the $\bar{x}$, the technique introduced in Section 2.3 can be used to derive the corresponding quadratic utility function at this point. This function effectively illustrates the robust design (quality) utility in a small neighborhood of $\bar{x}$, and could be used as an approximation of the efficient frontier to explore alternative solutions in the objective space. This eliminates the need of solving the problem again using a new preference structure, which is often a computationally expensive task for problems in a complex domain. The process will continue until a satisfactory solution is reached.

4. EXAMPLE PROBLEMS

In this section, a mathematical problem and the design of a two-bar structure are used to illustrate the validity of our proposed approach to the BORD problem.

4.1 A Mathematical Problem

The following mathematical problem is utilized to illustrate the tangible effects of our proposed approach.

$$\begin{aligned} &\text{minimize} \quad && f(x) = (x_1 - 4.0)^3 + (x_1 - 3.0)^4 + (x_2 - 5.0)^2 + 10.0 \qquad (4.1) \\ &\text{subject to} \quad && g(x) = -x_1 - x_2 + 6.45 \le 0 \\ &&& 1 \le x_1 \le 10 \\ &&& 1 \le x_2 \le 10 \,. \end{aligned}$$

The optimal solution of the above problem is located at the point x =(1.21280,5.23742), with f = -1.39378.

4.1.1 Robust Design Using the CP Method

The BORD problem for the mathematical problem is formulated as follows:

$$\begin{aligned} &\text{minimize} \quad && \left[\frac{\mu_f}{\mu_f^*}, \frac{\sigma_f}{\sigma_f^*}\right] \\ &\text{subject to} \quad && g(x) = -x_1 - x_2 + 6.45 + 2k\Delta x \le 0 \qquad (4.2) \\ &&& 1 + \Delta x \le x_1 \le 10 - \Delta x \\ &&& 1 + \Delta x \le x_2 \le 10 - \Delta x \,, \end{aligned}$$

where the mean function and the standard deviation can be derived using first-order Taylor expansion by considering the standard deviations of both x as Δx/3 :

$$\mu_f(x) = (x_1 - 4.0)^3 + (x_1 - 3.0)^4 + (x_2 - 5.0)^2 + 10.0 \tag{4.3}$$

$$\sigma_f(x) = \frac{\Delta x}{3}\sqrt{(3.0(x_1 - 4.0)^2 + 4.0(x_1 - 3.0)^3)^2 + (2.0(x_2 - 5.0))^2}\,. \tag{4.4}$$

To seek the ideal solutions, μ_f^* and σ_f^*, we solve the above optimization problem in Eqn. (4.2) by using Eqns. (4.3) and (4.4) separately as the design objective. When the size of variation is considered as $\Delta x = 1.0$ (Δx_1=Δx_2 =1.0) and the penalty factor k is taken as 1.0, the ideal solutions are obtained as $x^*_{\mu_f}$ =(2.00000,6.45074) for μ_f^*=5.10464 and $x^*_{\sigma_f}$ =(3.50559, 4.99187) for σ_f^*=0.416796. To solve the BORD problem using the CP approach based on the formulation in Eqn. (3.3), we choose $e_1 = e_2$=1.0 and the utopia point becomes u^*=(0.0,0.0).

A comparison is made for the solutions obtained from the WS method and the CP method. Fifteen evenly distributed combinations of w_1 and w_2 are considered, illustrating the robust design solutions under different preference structures. These solutions are plotted in the objective space formed by the normalized values of μ_f versus σ_f, as shown in Figures 5 and 6, respectively. From the results, we observe a tradeoff relationship between the mean μ_f and the standard deviation σ_f. We also observe that the results obtained from the WS method and the CP method are distinctively different for some of the weight settings. Reflected in Figure 6, the efficient solutions by using the WS method are all located at either end of the graph. There is a huge gap between these two clusters.

On the other hand, the solutions based on the CP method change gradually with the change of the weights and are evenly spread. The CP method can generate the efficient set which is shown to be a nonconvex curve as illustrated in Figure 6. The results we obtained for this mathematical problem clearly proves the advantage of the latter approach over the former in finding the nonconvex parts of the efficient set for robust design problems.

Insert Figure 5 Efficient Solutions Using the WS Method

Insert Figure 6 Efficient Solutions Using the CP Method

4.1.2 The Use of the Quadratic Utility Function

As discussed in Section 2.3, a decision making process of searching for the most preferred Pareto solution can be carried with the weighted Tchebycheff approach and the additional utility information at an efficient solution can be concurrently extracted from the related weighted-quadratic-sum problem. Based on Eqns. 2.11–2.15, for the mathematical example problem, the quadratic function has the following form:

$$q(F) = 0.5 a w_1^2 [f_1 - (w_2/w_1) f_2]^2 + p_1 f_1 + p_2 f_2 . \qquad (4.5)$$

For a given preference structure, say w_1=0.90 and w_2=0.10, the efficient solution is obtained as $(\bar{f}_1, \bar{f}_2) = (\bar{\mu}_f / \mu_f^*, \ \bar{\sigma}_f / \sigma_f^*)$=(0.99969, 6.80552) by using the CP method. Using Eqns. (2.13) and (2.15), Eqn. (4.5) becomes

$$q(\overline{F}) = 0.5 * a * 0.90^2[\overline{f_1} - (0.10/0.90)\overline{f_2}]^2 + 0.01098\overline{f_1} + 0.09878\overline{f_2}$$

$$= 0.024019\alpha + 0.682681. \qquad (4.6)$$

The above equation stands for a family of the quadratic functions at the considered efficient solution $\overline{F}$. The family is parametrized by α, a positive scalar that determines the steepness (or flatness) of the family functions in a neighborhood of $\overline{F}$. Each function represents the designer's utility in this neighborhood. According to Tind and Wiecek's theory (1997), the parameter α has to be large enough for the weighted-Tchebycheff problem and the quadratic-weighted sum problem to be related. Therefore there exists a minimum value α_{min} such that for every $\alpha \geq \alpha_{min}$ the existence of the quadratic functions is guaranteed. The value α_{min} yields the most flat quadratic function supporting the efficient frontier at $\overline{F}$ and therefore being the best one to inform the designer about the local tradeoffs. As Tind and Wiecek (1997) do not provide theoretical foundations on how to choose α_{min}, we are currently developing such a methodology. However, in this introductory paper we would like to better illustrate all the concepts. For example, for α = 0.47882, we get q($\overline{F}$)= 0.694434. Figure 7 depicts a plot of a portion of the efficient frontier in a neighborhood of $(\bar{\mu}_f/\mu_f^*, \bar{\sigma}_f/\sigma_f^*)$=(0.99969,6.80552) and the points determined by the quadratic utility function in Eqn. (4.6) for α = 0.47882 . In Table 1, a comparison between the efficient solutions obtained from the CP method and those derived using Eqn. (4.6) is provided. To derive the approximate efficient solutions from Eqn. 4.6, we use the same values of f_1 as those efficient solutions based on the CP method. The relative differences of f_2 are listed in the last column of Table 1. It is noted that the results are the same at the efficient solution for w_1=0.90 and w_2=0.10, a point for which the quadratic utility is derived. The largest error for f_2 is less than 3.35 % for the given approximation range.

Insert Figure 7 The Quadratic Utility Function (α=0.47882)

Insert Table 1 Estimation of the Efficient Solutions Using the Quadratic Utility Function q($\bar{F}$), change

4.2 Two-Bar Structure Design

A two-bar structure design problem is used to further illustrate the advantages of our approach. A schematic picture of this structure is illustrated in Figure 8. The objective of this design is to find two design variables: x_1, nominal diameter of the cross section, and x_2, the height of the two bar structure, so that the normal stress of each member is minimized. The whole design should be largely insensitive to the change of the two design variables. The design constraints are considered for the total volume, $g_1(x)$, and the buckling stress, $g_2(x)$. In the model, the external force Fr is 150 KN, the thickness of the cross section T is 2.5 mm, the width of the structure B is 750 mm, the Elastic Modules E is 210000 N/mm^2.

Insert Figure 8 The Two-Bar Structure

The original optimization problem of the two bar system is represented as

$$\text{minimize} \qquad f(x) = \frac{Fr}{2pTx_1x_2}\sqrt{B^2 + x_2^2}$$

$$\text{subject to} \qquad g_1(x) = \frac{2\pi T x_1 \sqrt{B_2 + x_2^2}}{700000} - 1.0 \le 0$$

$$g_2(x) = \frac{\dfrac{Fr}{2pTx_1x_2}\sqrt{B^2 + x_2^2}}{\dfrac{1}{8}p^2E\dfrac{T^2 + x_1^2}{B^2 + x_2^2}} - 1.0 \le 0 \qquad (4.7)$$

$$1.0 \le x_1 \le 100.0 \ \text{(mm)}$$

$$10.0 \le x_2 \le 1000.0 \ \text{(mm)}$$

The above formulation is transformed into a robust design problem when considering Δx_1=1.0, Δx_2= 5.0 and k=1.0. The ideal solution is identified as μ_f^* =330.359 (N/mm^2 and σ_f^* =2.54901 (N/mm^2). Robust design solutions with varying weighting factors (27 settings) are obtained based on the WS method and the CP method, respectively. The corresponding tradeoff plots between the mean value μ_f and the standard deviation σ_f are shown in Figures 9 and 10. It is noted that while the solution curve obtained from the WS method (Figure 9) is convex in general, the efficient solutions identified based on the CP method include several nonconvex regions as shown in Figure 10. This observation is confirmed by evaluating the Hessian matrix of the objective function, which is found to be semi negative definite. The two-bar problem further illustrates the limitation of the WS method for problems with a nonconvex efficient frontier.

Insert Figure 9 Efficient Solutions Using the WS Method (Two-Bar Problem)

Insert Figure 10 Efficient Solutions Using the CP Method (Two-Bar Problem)

4.3 How Are the Solutions Different from Those Based on S/N Ratio?

To illustrate the need of using the multiobjective approach to robust design, we compare the results obtained based on the CP method and those based on the signal-to-noise-ratio (S/N). The S/N ratio is a robust design criterion recommended by Taguchi. The smaller-the-better S/N ratio expressed in Eqn. (4.8) is based on the smaller-the-better loss function,

$$\eta = -10\log_{10}\left[\frac{1}{n}\sum_{i=1}^{n}(f_i(x)-\mu_f^*)^2\right]. \tag{4.8}$$

Here the ideal solution μ_f is used as the desired minimum value. In applying the S/N ratio, the performance attribute $f_i(x)$ fluctuates under the deviations of x. Nine points are picked over the grid formed by (x_1–Δx_1, x_1, x_1 + Δx_1) and (x_2–Δx_2, x_2, x_2 + Δx_2) to evaluate $f_i(x)$ under deviations. We use Δx_1=Δx_2=1.0 for the mathematical problem in Section 4.1 and Δx_1=1.0, Δx_2=5.0 for the two-bar problem in Section 4.2.

When using the S/N ratio as the criterion, the robust design optimum solution is obtained as (x_1,x_2) = (2.85368,5.59635) for the mathematical problem in Section 4.1 and (x_1, x_2) = (41.6357 mm, 721.377 mm) for the two-bar problem in Section 4.2. By comparing these solutions with those obtained based on the CP method, we find the optimal solution by the maximization of the S/N ratio in Eqn. 4.8 is close to the solutions generated by setting the weights as (0.65, 0.35) for the mathematical problem. For the two-bar problem, it is close to the weight setting of (0.85, 0.15). For both cases, the use of the S/N ratio is biased towards minimizing the mean of performance at the cost of increased performance deviations. On the other hand, the multiple objective approach to robust design offers more flexibility in making the tradeoffs between the two aspects of the robust design objective. The designer can then choose among a set of potential solutions based on his/her preference structure.

5. CONCLUSIONS

In this paper, we develop a robust design procedure that utilizes the recent developments on relating utility function optimization to CP method. The multiple aspects of the objective in robust design are addressed explicitly and designers are allowed to exercise their preference structures in capturing the quality utility. Compared to the existing methods for robust optimization such as the Taguchi's signal-to-noise ratio and the weighted-sum method, our approach has many advantages.

- *Capability of generating the efficient solutions*: Based on the discussion in Section 2.1 and the two example problems demonstrated in Sections 4.1.1 and 4.2, we recommend that the specific CP approach, the Tchebycheff method, be used to determine the robust design solution guaranteed to be the one that belongs to the set of efficient solutions. These solutions illustrate the tradeoffs of achieving the desired performance and its robustness. This is especially critical when the efficient solutions form a nonconvex frontier, in which case the weighted-sum method may fail.

- *Capability of measuring utility:* By utilizing the recent developments on relating utility function optimization to the CP method, we illustrate that these two methods could complement each other. While the CP method is used as a tool for finding a solution, a quadratic utility function is generated to specify a decision maker's utility function in a neighborhood of this solution (Sections 2.3 and 4.1.2). This overcomes the limitation of optimizing nonlinear utility function directly over the efficient set that is often computationally complex. On the other hand, the quadratic quality utility obtained at the solution point can be used as an approximation of the efficient frontier to explore the solutions in that neighborhood.
- *Interactive robust design procedure:* We develop a robust design procedure that is interactive and allows the designer to exercise different preference structures until a satisfactory robust design solution is obtained. Constructing the utility function does not require any additional information that is needed to perform the Tchebycheff approach and to find a Pareto solution. Therefore, a decision making process of searching for the most preferred Pareto solution can be carried with the weighted-Tchebycheff approach and additional utility information can be concurrently extracted from the related quadratic weighted-sums problem.
- *Significance of the multiobjective optimization approach to robust design*: We illustrate that by using Taguchi's S/N ratio as a single criterion for robust design, designers are not able to explicitly address the tradeoff between

achieving the design performance and its robustness. For the two example problems, the results indicate that the robust design solutions obtained using the S/N ratio is biased towards minimizing the mean of performance at the cost of increased performance deviations. The CP method appears to offer important advantages over the Taguchi's S/N ratio. The developed multiple objective approach to robust design offers more flexibility in addressing the multiple aspects of robust design.

In this paper, for illustration purposes, we consider single performance attribute in both example problems. It should be noted that the principles illustrated can be extended to robust design problems with multiple performance attributes or any multiobjective optimization problems. In the case that the robust design problem has multiple performance attributes, we could either develop two aggregate objective functions, one for the mean of performance and the other for the performance deviations, and treat the problem as a BORD problem; or we could treat the multiple objectives individually and solve the problem as a multiobjective optimization problem. The principle of generating the quadratic utility function at an efficient solution can be extended to more than two objectives. In terms of the future work in this area of research, we plan to conduct the refinement of the choice of the parameter α of the quadratic utility function. The applications of our approach to complex engineering systems are also under consideration.

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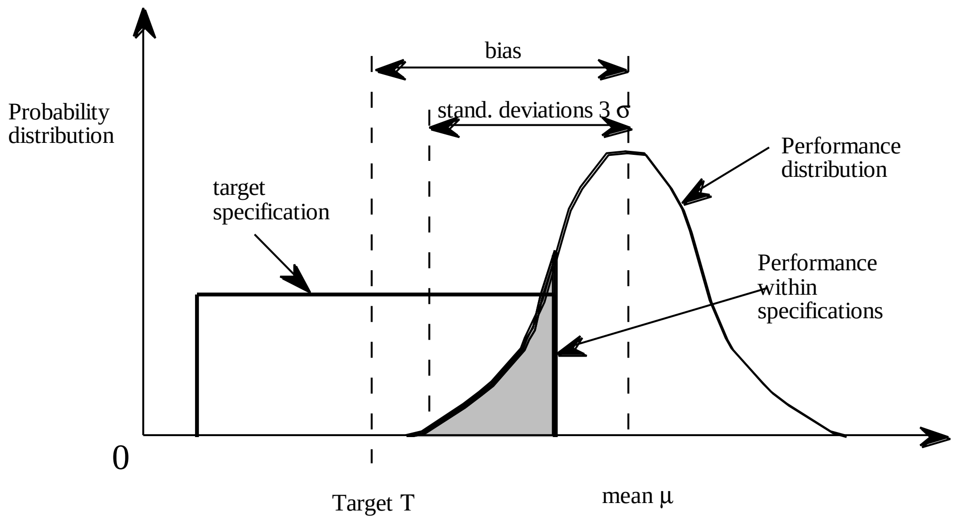

Figure 1 Quality Distribution in Robust Design

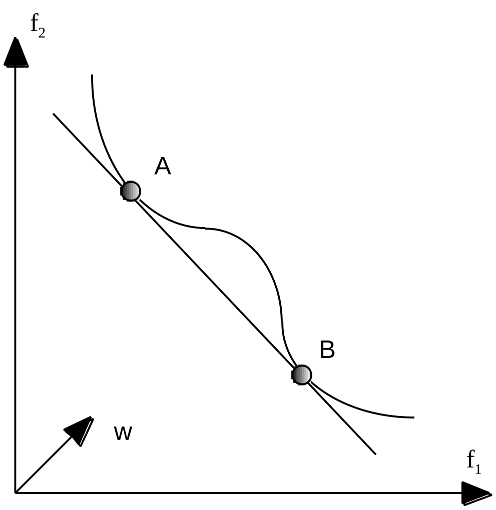

Figure 2 Generating Efficient Solutions by the WS method

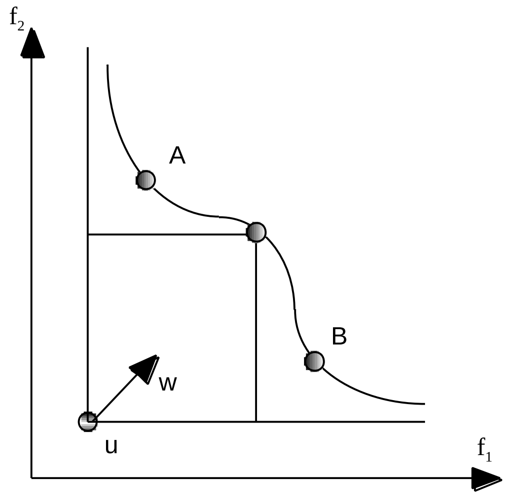

Figure 3 Generating Efficient Solutions by the Weighted-Tchebycheff Method

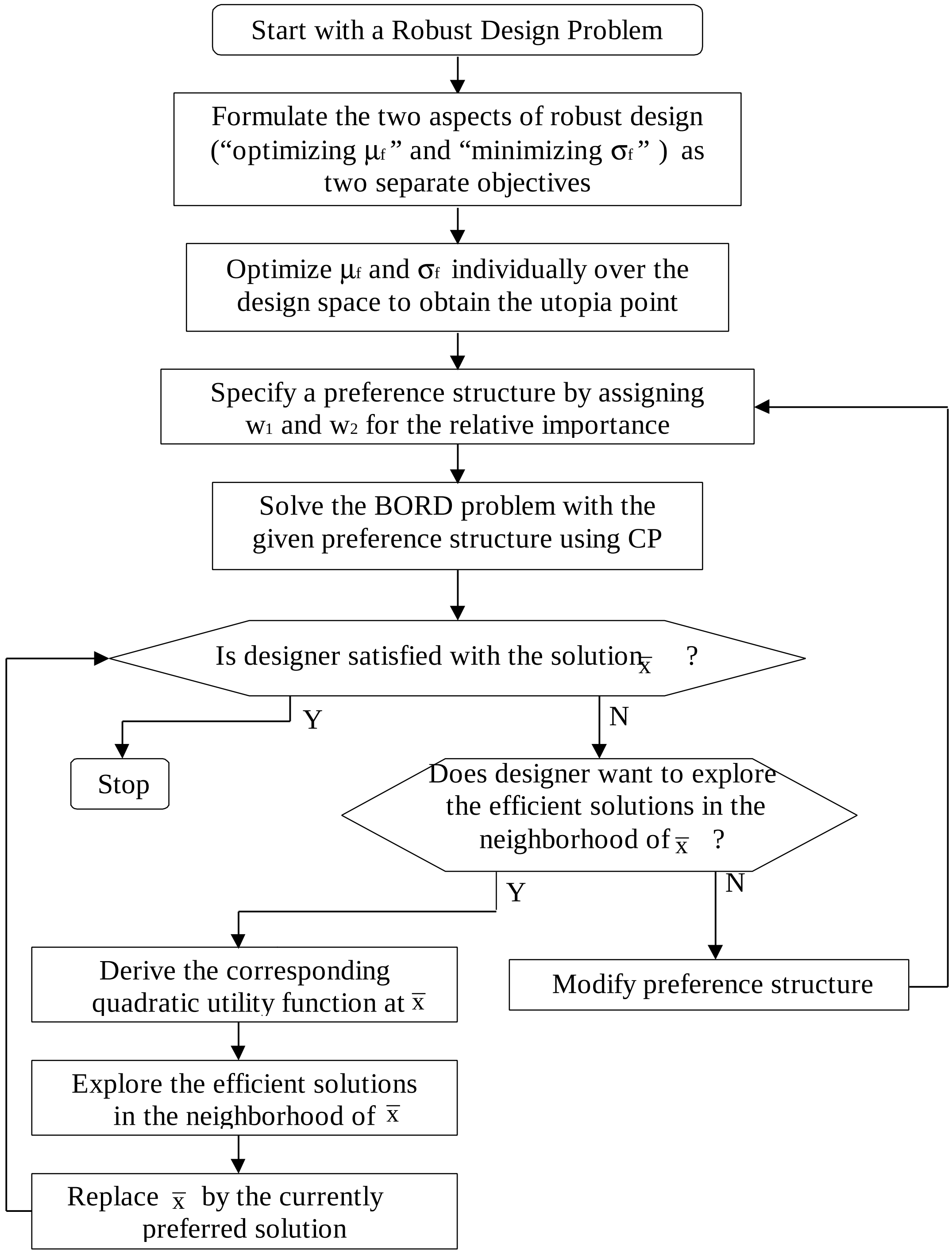

Figure 4 A Robust Design Procedure for Addressing Multiple Aspects of Robust Design

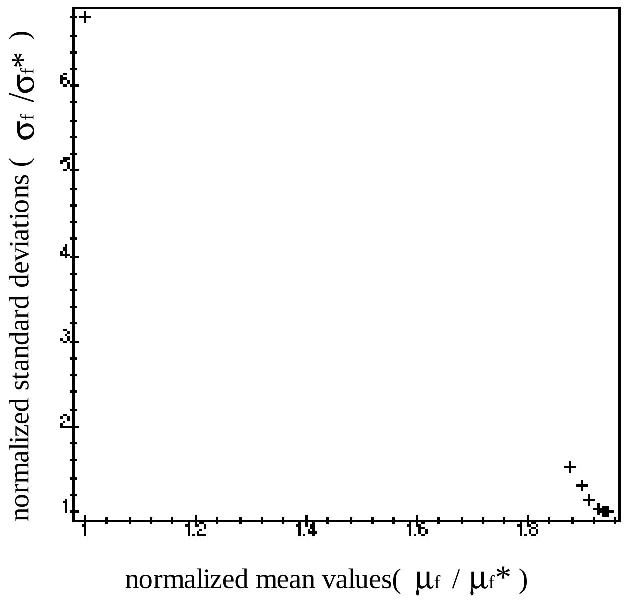

Figure 5 Efficient Solutions Using the WS Method

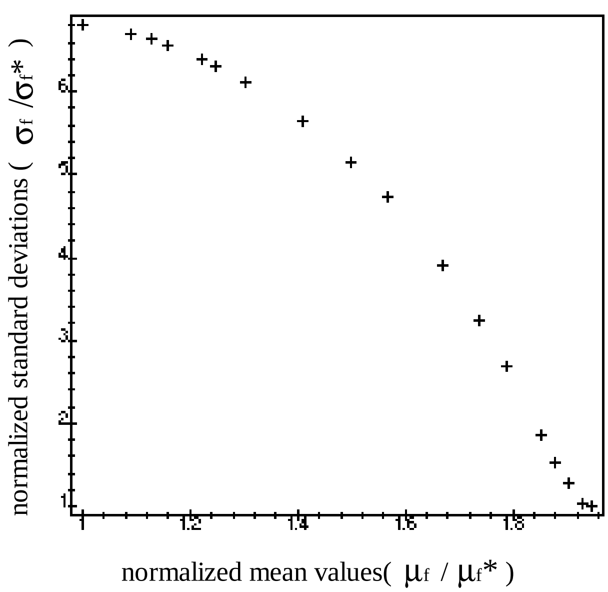

Figure 6 Efficient Solutions Using the CP Method

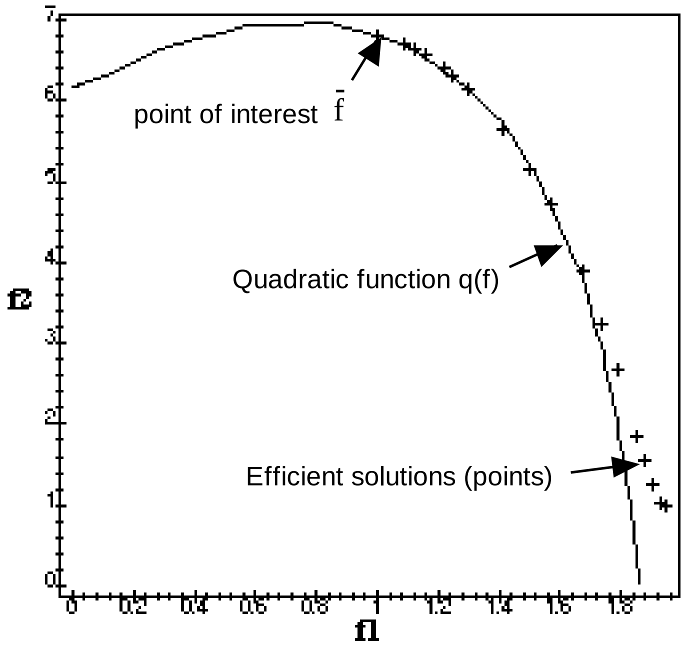

Figure 7 The Quadratic Utility Function (a=0.47882)

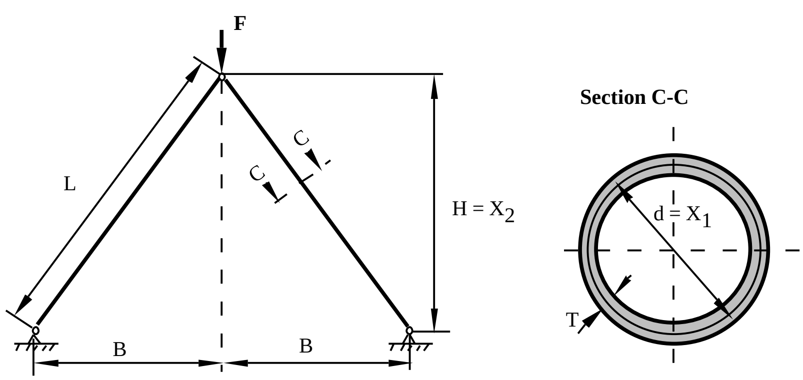

Figure 8 The Two-Bar Structure

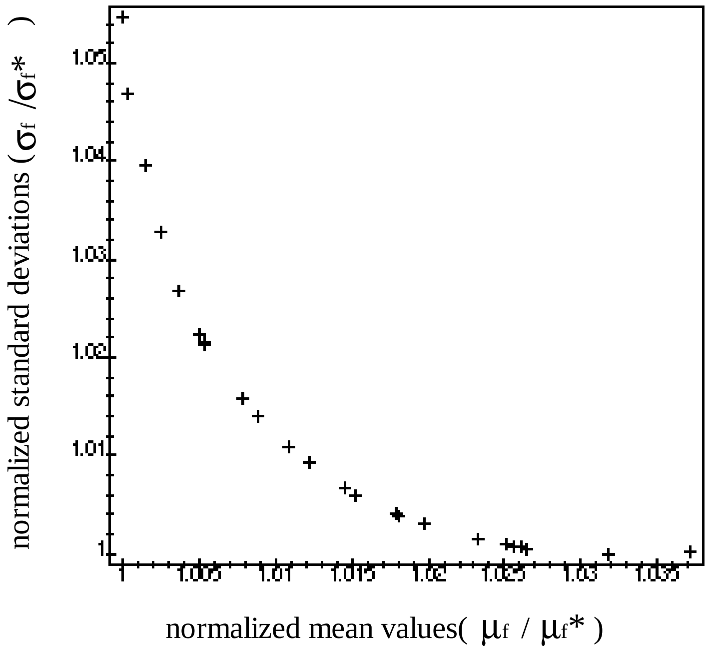

Figure 9 Efficient Solutions Using the WS Method (Two-Bar Problem)

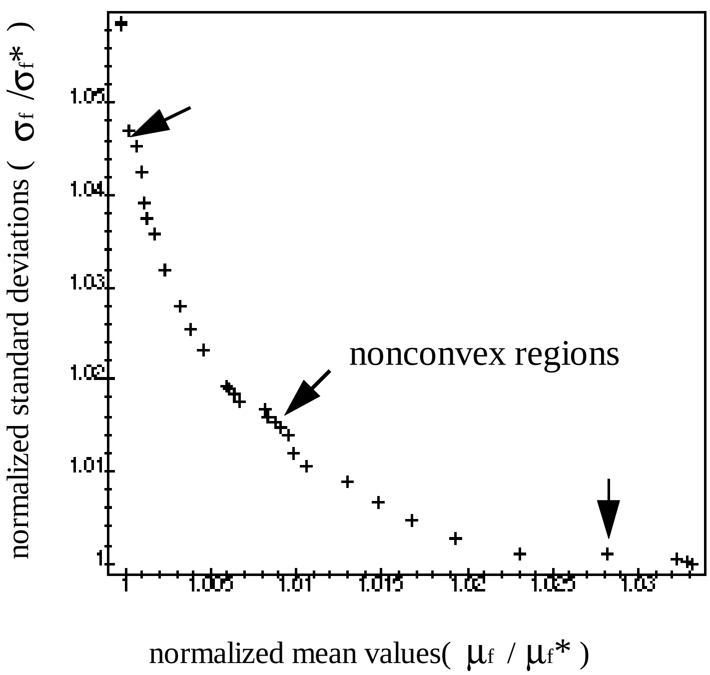

Figure 10 Efficient Solutions Using the CP Method (Two-Bar Problem)

Table 1 Estimation of the Efficient Solutions Using the Quadratic Utility Function q(F)

Efficient solutions obtained using CP	Solutions with f_2 estimated by q(F)	Relative difference of f_2 (%)
(1.67096, 3.89996)	(1.67096, 3.76941)	3.34738
(1.56610, 4.72223)	(1.56610, 4.73626)	0.29717
(1.41002, 5.64496)	(1.41002, 5.68802)	0.76281
(1.30102, 6.12612)	(1.30102, 6.13554)	0.15370
(1.24947, 6.31014)	(1.24947, 6.30194)	0.13000
(1.15925, 6.56689)	(1.15925, 6.53611)	0.46867
(0.99969, 6.80552)	(0.99969, 6.80552)	0.00000