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A WEIGHTED THREE-POINT-BASED STRATEGY FOR VARIANCE ESTIMATION

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ABSTRACT

In manufacturing processes, it is widely accepted that uncertainty plays an important role and should be taken into account during analysis and design processes. However, uncertainty quantification of its effects on an end-product is a very challenging task, especially when an expensive computational effort is already needed in deterministic models such as sheet metal forming simulations. In this paper, we focus our work on the variance estimation of the system response. A weighted three-point-based strategy is proposed to efficiently and effectively estimate the variance of the system response. Three first-order derivatives for each variable are used to estimate the nonlinear behavior and variance of the system. The details of the derivation of the approach are presented in the paper. The optimal locations of the three points along each axis in the standard normal space and weights for input variables following normal distributions are proposed as (-1.8257,0.0,+1.8257) and (0.075,0.850,0.075), respectively. For input variables following uniform distributions *U*(-1,1), the optimal locations and weights are proposed as (-0.84517, 0.0,+0.84517) and (0.04667,0.90666,0.04667), respectively. The proposed approach is applicable to nonlinear and multivariable systems as well as problems having no explicit function such as those design simulations based on finite element methods. The significant accuracy improvement over the traditional first-order approximation is demonstrated with a number of test problems. The proposed method requires significantly less computational effort compared with the Monte Carlo simulations. Discussions and conclusions of this work are given at the end of the paper.

Key words: weighted three-point-based strategy; first order approximation; uncertainty propagation; variance estimation; design under uncertainty.

1. INTRODUCTION

Uncertainty analysis has been an important subject in quantitative system analysis and design due to the inherent uncertainty embedded in systems [Law and Kelton, 1991; Ross, 1993; Draper, 1995; Zhao and Ono, 1999; Du and Chen, 2000; Thronton, 2000; Du and Chen, 2001; Lee and Park, 2001; Du and Chen, 2002; Oberkampf *et al*., 2002]. For most engineering applications, mean and variance of the system response are commonly regarded as the most important parameters of interest for characterizing a probabilistic system. In general, it is known that the variance estimation is a challenging task, especially when its corresponding deterministic model is very costly.

In an exact fashion, the variance is directly calculated by the integration of the product of the deviation squared and its probability density function. Unfortunately, the exact approach is not always applicable and practical to conduct the estimation. An alternative powerful technique for the uncertainty propagation is the Monte Carlo Simulation (MCS). However, this approximation technique requires a significant amount of computational effort, which is generally viewed as an impractical solution for many problems, particularly when it is used with a simulation model requiring a huge computational effort for each run like sheet metal forming simulations [Cao *et al*., 2000]. One solution is to utilize response surface methodology [Koc *et al*., 2000; Varadarajan *et al*., 2000; Buranathiti *et al*., 2002; Baghdasaryan *et al*., 2002; Gu and Yang, 2003; Helton and Davis, 2003; Buranathiti *et al*., 2004]. However, a number of samples from a costly model are needed to create a good approximation. In addition to MCS, an arguably popular and efficient method for the variance estimation is the first order approximation [Du and Chen, 2000; Thronton, 2000]. However, the generality and accuracy of this method are largely compromised by the simplicity of the method.

Therefore, there is a need for an efficient and still accurate approach for uncertainty propagation in general systems, especially in design of those systems that have to be modeled by numerical techniques (e.g. finite element methods) and treated as a black box. For specific problems, approximated stochastic models may be obtained by random field finite element [Liu *et al*., 1986], variational approach [Liu *et al*., 1988], interval analysis [Qiu and Elishakoff, 1998; Braibant *et al*., 1999; Nakagiri and Suzuki, 1999], spectral approach [Ghanem and Spanos, 2003], and so on.

In this paper, we propose an efficient and effective approach for the variance estimation of a general system response. We named this approach as 'weighted three-point-based strategy', which is a significant extension of the traditional first order approximation. This work reviews a number of uncertainty propagation techniques in Section 2. The proposed weighted three-point-based strategy is presented in Section 3. The derivation of the optimal parameters (i.e. locations and weights) is shown in Section 4 followed by the demonstration of the effectiveness of this approach in Section 5. The discussions and conclusions of this work are presented in Section 6.

2. UNCERTAINTY PROPAGATION TECHNIQUES

Uncertainty propagation is a process that propagates the effect of the uncertainty of input variables on a system response, described by the statistical characteristics of the response. The mean estimation in general is relatively simpler than the variance estimation. In this work, our focus is on the variance estimation of a system response.

In order to estimate the variance, the classical probability technique using the exact integration with the probability density function is the most effective to obtain the exact prediction. Unfortunately, it is usually not applicable. Monte Carlo simulation, a simulation-based technique, is an alternative and powerful technique. We consider this approach as an extreme edge of the approximation techniques having the greatest accuracy and computational requirement. Another extreme edge of approximation techniques is the first order approximation, which is probably the most efficient techniques for the variance estimation.

The selection of approximation techniques is based on a combination of the applicability of the approximation technique, the computational effort needed, and the prediction accuracy. A review of uncertainty propagation techniques is discussed in the rest of this section followed by the presentation of our proposed weighted three-point-based strategy in the next section.

2.1. Exact Approach

The exact approach for the variance estimation is an integration of a product of the function and the corresponding probability density function. As a result, the variance of a system is exactly derived. The variance of a random variable x is defined as

$$\mathrm{VAR}[x] = \mathrm{E}\left[(x-\mathrm{E}[x])^2\right] = \int_x (x-\mathrm{E}[x])^2 f(x)dx \,, \qquad (1$$

where $\mathrm{VAR}[\cdot]$ is the variance operator, $\mathrm{E}[\cdot]$ is the expected operator, and $f(x)$ is the probability density function of the random variable x. The expected value of x is defined as

$$\mathrm{E}[x] = \int_x x f(x)dx \,. \qquad (2$$

Similarly, we obtain the variance of the function $g(x)$ as

$$\mathrm{VAR}[g(x)] = \int_x (g(x)-\mathrm{E}[g(x)])^2 f(x)dx \,. \qquad (3$$

For a limited number of cases in engineering design problems, we can obtain the variance of a system by using the exact approach. Unfortunately, this approach is not always possible for most cases, especially if there is no explicit analytical function. In addition, the variance operator is not a linear operator, which makes the function manipulation on the variance operator become very impractical. Therefore, other approximation techniques are often used as practical alternatives to the exact approach.

2.2. Monte Carlo Simulation

Monte Carlo simulation (MCS), a simulation-based approach, is a widely used approximation technique based on pseudo random numbers. It is generally considered to be the most powerful approach for uncertainty propagation but nevertheless the most expensive approach. Details and applications can be consulted in literature [Law and Kelton, 1991; Ross, 1993]. The keys of this approximation technique are the random number generator and the resource availability. In engineering applications, significant computational effort is reduced by using response surface methodology to substitute costly original models like finite element methods [Koc *et al*., 2000; Varadarajan *et al*., 2000; Buranathiti *et al*., 2002; Baghdasaryan *et al*., 2002; Gu and Yang, 2003].

Although MCS can provide satisfactory prediction if the sample size is large enough, one major concern from MCS with a small number of samples is the repeatability or the robustness of the prediction. Considering a quadratic function ($y=0.5x^2+x$) with x follows a normal distribution with a mean of 1.0 and variance of 0.2, the repeatability of the solutions using MCS with 300 samples and more is shown in Fig. 1. It is observed that the results are not reliable nor repeatable when the magnitude of the number of samples is less than $O(10^4)$.

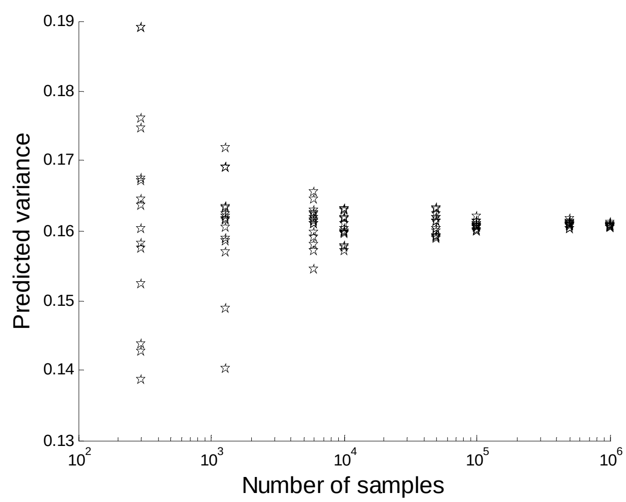


Figure 1. The convergence of solutions from MCS in a semi-log scale.

2.3. First Order Approximation

One of the most efficient techniques for uncertainty propagation is the first order approximation. Generally, we assume that each variable is statistically independent to each other and there is no implicit relation among variables (independent variables). The technique used is based on a linearized model of variation using Taylor's expansion [Du and Chen, 2000; Thronton, 2000; Lee and Park, 2001; Suri *et al*., 2001; Bakr and Butler, 2002; Du and Chen, 2002]. We begin with the Taylor's expansion of a response function y around $\mathbf{x} = \bar{\mathbf{x}}$ ($\bar{\mathbf{x}}$ represents the nominal value) as follows

$$y \approx y(\bar{\mathbf{x}}) + \sum_{i\in I} \frac{\partial y}{\partial x_i} dx_i + \frac{1}{2}\sum_{i\in I} \frac{\partial^2 y}{\partial x_i^2}(dx_i)^2 + \sum_{i\in I}\sum_{j\in J, i\neq j} \frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j} dx_i dx_j + \cdots . \quad (4$$

Often the higher order terms are neglected to simplify the expansion into

$$y \approx y(\bar{\mathbf{x}}) + \sum_{i\in I} \left\{ \frac{\partial y}{\partial x_i} \Delta x_i \right\} . \quad (5$$

Let dx_i represent the infinitesimal distance of the expansion around $\mathbf{x} = \bar{\mathbf{x}}$ as

$$x_i = \bar{x}_i + dx_i . \quad (6$$

The variance operator on Eq. (6) leads to the expression of the variance of the dx_i part as

$$\mathrm{VAR}[dx_i] = \mathrm{VAR}[x_i - \bar{x}_i] = \mathrm{VAR}[x_i] , \quad (7$$

where the variance term of the constant term $\bar{x}_i$ vanishes. In order to estimate the variance of y, we take the variance operator on Eq. (5) and obtain an approximated variance of y as follows

$$\mathrm{VAR}[y] = \sum_{i\in I} \left\{ \left(\frac{\partial y}{\partial x_i}\right)^2 \mathrm{VAR}[x_i] \right\} . \quad (8$$

Note that the variance of the constant term $y(\bar{\mathbf{x}})$ varnishes.

In many cases where there is no explicit response function, one way to obtain the system sensitivity is to utilize a finite difference technique to obtain the sensitivity of system or derivative as follows

$$\frac{\partial y}{\partial x_i} \approx \frac{y(\bar{\mathbf{x}} + \Delta x_i) - y(\bar{\mathbf{x}})}{\Delta x_i} . \quad (9$$

The first order approximation requires (2n) function evaluations for an n-variable system using the finite difference calculation.

This approximation method is efficient and effective for functions with low nonlinearity and/or low variation. When a large variation is taken into account and/or a system function is highly nonlinear, the approximation may significantly deviate from the exact solution. A comparison will be shown in Section 5. From the need of an accurate and efficient approximation method for general applications, the extension of the first order approximation is proposed in the next section.

3. WEIGHTED THREE-POINT-BASED STRATEGY

In this work we propose an effective and still efficient method for the uncertainty propagation to estimate the response variance of a design system. The proposed approach still holds the computational efficiency of the traditional first-order approximation but with much improved accuracy. We begin the derivation with expression used in the Taylor's first order expansion of a response function y around $\mathbf{x} = \bar{\mathbf{x}}$ in Eq. (5) and the approximated variance of the system in Eq. (8).

We propose a weight-estimation scheme with the multiple-point-based technique to increase the accuracy of the variance prediction. The basic idea is to determine the best way to obtain the additional samples for calculating the derivative, $\frac{\partial y}{\partial x_i}$, of the response with respect to the i^{th} variable x_i. Instead of considering only derivatives at the nominal point, we sample more observations to calculate the derivatives, $\left.\frac{\partial y}{\partial x_i}\right|_{\bar{x}_{k\neq i}, x_i^j}$, at nearby points around the nominal. We also believe that the influence contributed from the additional observations should not have the same weight as that from the nominal value. Therefore, the total variance of a system response is then formulated as follows

$$\mathrm{VAR}[y] = \sum_{i\in I} \left[\mathrm{VAR}[x_i] \cdot \sum_{j\in J} \left\{ \phi_j \left(\left.\frac{\partial y}{\partial x_i}\right|_{\bar{x}_{k\neq i}, x_i^j} \right)^2 \right\} \right] , \quad (10$$

where ϕ_j is the corresponding weight to the j^{th} point. The next major question is how to best select these additional locations and the best corresponding weights so that the influence of these additional observations best increases the overall approximation accuracy. The derivation of these values is presented in the next section.

4. DETERMINATION OF THE OPTIMAL COMBINATION OF WEIGHTS AND OBSERVATION LOCATIONS

We propose a framework for the determination of the optimal combination of weights and observation locations to best estimate the variance of system responses at a low computational cost. The work is formulated based on a three-point-based approximation. The essence of the method is to try to find the best way to use two more first-order derivatives around the nominal design point on both lower and upper sides to best describe basis functions.

To define the multiple observation locations, we designate $x_i^{(j)}$ as the j^{th} point of the i^{th} variable x_i and write the expansion of x_i around the based point $\bar{x}_i$ as follows

$$x_i^{(j)} = \bar{x}_i + \Delta x_i^{(j)} , \quad (11$$

where $\Delta x_i^{(j)}$ is the distance between the j^{th} point $x_i^{(j)}$ and its based point $\bar{x}_i$. In this work, we consider the nominal value of the i^{th} variable x_i as the base point $\bar{x}_i$.

We impose the following restrictions on how to select the observation locations and the corresponding weights so that the formulation is practical and efficient to be implemented. First, we define the convention of node numbering, i.e., j = 1, 2, and 3, as shown in Fig. 2. Second, let the middle point be at the mean value, i.e., $x_i^{(2)} = \bar{x}_i$. This restriction is due to the need to study the behavior at the nominal value in practice. Third, we impose a symmetry condition of the observation locations as follows

$$\Delta x_i^{(2)} = 0 , \text{ (i.e. } x_i^{(2)} = \bar{x}_i \text{)} \quad (12$$

$$\Delta x_i^{(1)} = -\Delta x_i^{(3)} , \quad (13$$

$$\Delta x_i^{(3)} > 0 . \quad (14$$

A convention of the numbering and the illustration of the approach are shown in Fig. 2.

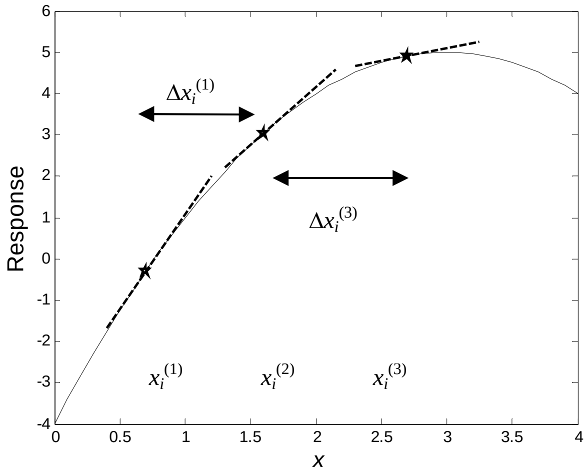


Figure 2. The illustration of the approach and the convention of node numbering.

Finally, we impose a conservation condition of weights ϕ_j in Eq. (10) as follows

$$\sum_{j \in J} \phi_j = 1 = \phi_1 + \phi_2 + \phi_3 , \quad (15$$

and $\phi_1 = \phi_3$. (16

Therefore, the weights and locations of observation points can be defined by using two parameters, $\Delta x_i = \Delta x_i^{(3)}$ and ϕ_1. In the rest of this session, we illustrate the methodology of determining these two parameters when x follows a normal distribution or a uniform distribution.

4.1. Normal Distribution

To determine the optimal combination of weights and point locations, we consider the normal distribution $f(x)$, the most widely used probability density function, as follows

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} , \quad (17$$

where μ is the expected or mean value of x, and σ^2 is the variance of x. For simplicity of the formulation, we transform the random variable x into the standard normal space z, i.e. $z \in (-\infty,\infty)$, by using the following standard normal expression

$$z = \frac{x-\mu}{\sigma} . \quad (18$$

Therefore, the mean and variance of z are 0 and 1, respectively. Now we need two additional equations to complete the system equation to determine the optimal combination of weights and point locations. We utilize two polynomial basis functions to represent general systems of the random variable z as follows

$$g_1(z) = z^2 , \quad (19$$

and

$$g_2(z) = z^3 . \quad (20$$

The two polynomial basis functions are selected to capture the nonlinearity of a general function at around the nominal point, a similar concept to the derivation of Gauss quadrature in numerical integration methods. It should be noted that we did not put arbitrary constants on the reference basis functions for brevity because the constants will cancel each other at the latter stages.

Let σ_{gi}^2 be the exact variance of the function g_i, and $\tilde{\sigma}_{gi}^2$ be the approximated variance of the function g_i using the weighted three-point-based strategy. The optimal combination of ϕ_j is determined by selecting a set resulting that $\tilde{\sigma}_{gi}^2$ is equal or the closest to σ_{gi}^2. Using the exact approach in Eq. (3), we calculate the exact variances of these two functions as follows

$$\mathrm{VAR}[g_1(z)] = \sigma_{g1}^2 = 2 , \quad (21$$

$$\mathrm{VAR}[g_2(z)] = \sigma_{g2}^2 = 15 . \quad (22$$

Using the weighted three-point-based strategy in Eq. (10), we write the approximated variance $\tilde{\sigma}_{g1}^2$ and $\tilde{\sigma}_{g2}^2$ as follows

$$\tilde{\sigma}_{g1}^2 = 8\sigma_z^2 \phi_1 (\Delta z)^2 , \quad (23$$

$$\tilde{\sigma}_{g2}^2 = 18\sigma_z^2 \phi_1 (\Delta z)^4 . \quad (24$$

By equating Eq. (21) with Eq. (23), and Eq. (22) with Eq. (24), i.e.,

$$\sigma_{g1}^2 = \tilde{\sigma}_{g1}^2 , \quad (25$$

$$\sigma_{g2}^2 = \tilde{\sigma}_{g2}^2 . \quad (26$$

The optimal combination of weights and observation locations can be obtained as

$$(\Delta z)^2 = \frac{\sigma_{g2}^2}{\sigma_{g1}^2} \frac{8}{18} , \quad (27$$

and

$$\phi_1 = \frac{(\sigma_{g1}^2)^2 18\sigma_z^2}{\sigma_{g2}^2 (8\sigma_z^2)^2} . \quad (28$$

For the given conditions, we obtain the following design values for the weighted three-point-based strategy:

$$\Delta z = \frac{\sqrt{30}}{3} , \quad (29$$

$$\phi_1 = \frac{3}{40}, \phi_2 = \frac{17}{20}, \phi_3 = \frac{3}{40} . \quad (30$$

The optimal values, calculated from the above theoretical analysis, of the locations of samples and their corresponding weights in the weighted three-point-based approximation in Eq. (10) are summarized in Table 1.

Table 1. Summary of the parameters for the weighted three-point-based strategy for inputs following normal distributions.

Sampling location	Weight
$z_1 = -1.8257$	$\phi_1 = 0.075$
$z_2 = \;\; 0.0000$	$\phi_2 = 0.850$
$z_3 = +1.8257$	$\phi_3 = 0.075$

Note that these parameters are corresponding to the standard normal z space; general cases can be calculated by transforming the variables back to the original space in Eq. (18).

4.2. Uniform Distribution

The approach presented above in the previous section can also be used for distributions other than the normal distribution. Here, we briefly present a solution for the uniform distribution. For uniform distributions: $x \in [x_{min}, x_{max}]$, the probability density function $f(x)$ is defined as

$$f(x) = \frac{1}{x_{max} - x_{min}} . \quad (31$$

For simplicity of the formulation, we transform the random variable x into a master space z (i.e. $z \in [-1,1]$) by using the following linear mapping

$$z = 2\frac{x - x_{min}}{x_{max} - x_{min}} - 1 . \quad (32$$

Therefore, the mean and variance of z are 0 and 1/3, respectively. We followed the same procedure as previously shown. For the uniform distribution, we calculate the exact variances of the two functions as follows

$$\mathrm{VAR}[g_1(z)] = \sigma_{g1}^2 = \frac{4}{45} , \quad (33$$

$$\mathrm{VAR}[g_2(z)] = \sigma_{g2}^2 = \frac{1}{7} . \quad (34$$

For the given conditions, we obtain the following design values for the weighted three-point-based strategy (Table 2):

$$\Delta z = \frac{\sqrt{35}}{7} = 0.84515 , \quad (35$$

$$\phi_1 = \frac{7}{150}, \phi_2 = \frac{68}{75}, \phi_3 = \frac{7}{150} . \quad (36$$

We use these parameters in the master space [-1,1], and then transform the system variables back to the original space for further calculations.

Table 2. Summary of the parameters for the weighted three-point-based strategy for inputs following uniform distributions.

Sampling location	Weight
$z_1 = -0.84517$	$\phi_1 = 0.04667$
$z_2 = \ \ 0.00000$	$\phi_2 = 0.90666$
$z_3 = +0.84517$	$\phi_3 = 0.04667$

The flow chart of the weighted approach is summarized in Fig. 3.

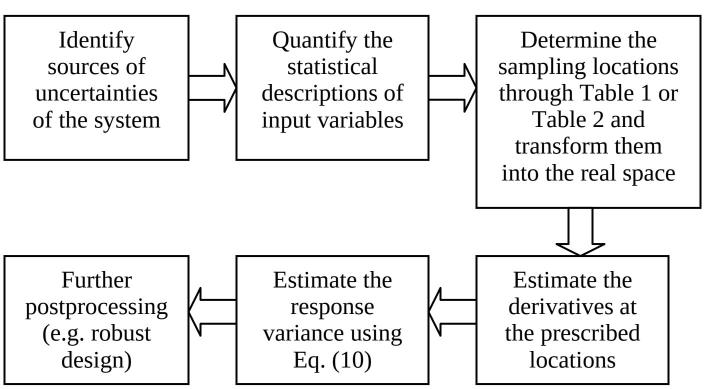


Figure 3. The flow chart of the weighted three-point-based strategy.

5. ILLUSTRATION EXAMPLES

The following demonstration examples are used to illustrate the improvement of the new enhanced technique for the variance estimation (uncertainty propagation). Given the following test functions:

1. $y = 0.5x^2 + x$,
2. $y = 0.2x^3 + 1.5x^2 - 0.5x$,
3. $g(\mathbf{x}) = 0.5x_1^2 - x_1 + 0.7x_2^2 - 0.5x_2$,
4. $g(x) = 4\sin(x + 0.5)$,
5. $g(x) = 2\exp(0.65x - 0.3)$,
6. $g(\mathbf{x}) = x_1^2 - 0.5x_1 + 2x_2^2 - 0.5x_2 + 0.5x_1^2 x_2 + 0.05x_1^2 x_2^2$,
7. $g(\mathbf{x}) = \ln(10 - x_1) + 2\ln(9 - x_2) + \ln(8 - x_3) + x_4^2 + 2\ln(7 - x_5) + 0.75x_6^2$,
8. $g(\mathbf{x}) = \ln(10 - x_1) + 2\ln(9 - x_2) + \ln(8 - x_3) + x_4^2 + 2\ln(7 - x_5) + 0.75x_6^2 + 0.5x_2x_4 + 0.5x_3x_5^2$.

The variances of the system responses calculated from the referred approximation techniques are summarized in Table 3 and Table 4 for random variables following normal distributions and uniform distributions, respectively. A ratio of 1 indicates the exact match to the exact value obtained by using Eq. (3).

It should be noted that for the column of 'Equally weighted 3 points,' we refer to the weighted three-point-based approximation that uses the same weight ($\phi = 1/3$) for all three points. The coefficient of variance (COV = $\bar{x}/\sigma$) can be determined directly since the mean values of input variables are mostly one. In addition, the ratio, which is closer to 1.0, indicates a better match to the exact value.

We observe that the proposed method shown in the last column can well predict the variance for the test functions in Table 3 for the input variables following normal distributions and in Table 4 for the input variables following uniform distributions. If the exact solution is available, the predictions from other techniques are shown in ratios compared with the exact solution. The overall score from the proposed method is much better than that from the traditional first order approximation while the computational effort from the proposed method is much less than that from MCS. The average performance of the proposed method is better than 97%.

The nonlinearity of the test problems is examined in Fig. 4. The problems are shown with approximated cubic polynomial functions associated with different fitted ranges. We increase the range for approximated functions 1, 2, 3, respectively. It is obvious that the smaller covered region (or variation in this study) results a better approximation. If a large variation presents, more observation points and/or higher order basis functions in Eqs. (19-20) may be needed for better approximation. In literature [Koc *et al.*, 2000; Varadarajan *et al.*, 2000; Buranathiti *et al.*, 2002; Baghdasaryan *et al.*, 2002; Gu and Yang, 2003], many engineering problems can be well surrogated by using polynomial functions.

Table 3. The predictions of variance for different test problems obtained from different techniques by assuming all random variables follow the given normal distributions.

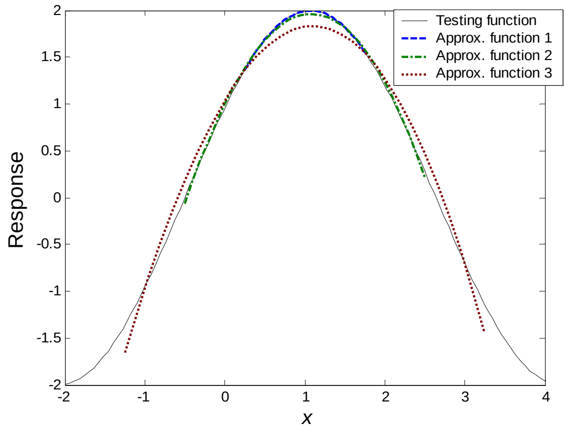
Given	$N(\mu,\sigma)$[§]	Response variance by Exact approach	Ratio of the estimated value to the exact value ** MCS (1M)	First order approximation One pt.	Equally weighted 3 pt.	Optimally weighted 3 pt.
(1)	(1.0,0.25)	0.2520	0.9990	0.9922	1.0267	1.0000
	(1.0,0.75)	2.4082	0.9969	0.9343	1.2263	1.0000
	(2.0,0.75)	5.2207	1.0010	0.9697	1.1044	1.0000
	(1.0,1.00)	4.5000	0.9956	0.8889	1.3827	1.0000
(2)	(1.0,0.25)	0.6498	1.0022	0.9244	1.2108	0.9888
	(1.0,0.50)	3.1956	1.0001	0.7518	1.6932	0.9636
	(2.0,0.50)	17.1156	0.9972	0.9116	1.2276	0.9827
	(1.0,0.75)	9.4801	1.0030	0.5702	2.2045	0.9379
(3)	(1.0,0.25)	0.0564	0.9963	0.8975	1.3530	1.0000
	(1.0,0.50)	0.2950	1.0014	0.6864	2.0800	1.0000
	(2.0,0.50)	1.6650	1.0005	0.9444	1.1914	1.0000
	(1.0,0.75)	0.9239	0.9990	0.4932	2.7458	1.0000
(4)	(1.0,0.25)	0.0339	0.9993	0.1475	3.9270	0.9979
	(1.0,0.50)	0.4052	0.9992	0.0494	4.1283	0.9672
	(2.0,0.50)	2.1605	0.9999	1.1883	0.9691	1.1390
	(1.0,0.75)	1.5003	1.0019	0.0300	3.8306	0.8851
(5)	(1.0,0.25)	0.2213	1.0009	0.9611	1.0773	0.9873
	(1.0,0.50)	0.9973	0.9969	0.8531	1.3028	0.9543
	(2.0,0.50)	3.6595	1.0021	0.8531	1.3028	0.9543
	(1.0,0.75)	2.7407	1.0009	0.6985	1.6529	0.9132
(6)	(1.0,0.25)	1.5520	0.9984	0.9491	1.0969	0.9824
	(1.0,0.50)	7.1768	0.9967	0.8210	1.3323	0.9361
	(1.0,0.75)	19.9424	0.9981	0.6648	1.5963	0.8744
(7)*	(1.0,0.25)	N/A	0.4160	0.9700	1.1007	0.9994
	(1.0,0.50)	N/A	1.8127	0.8904	1.3704	0.9984
	(1.0,0.75)	N/A	4.6190	0.7863	1.7399	1.0008
(8)*	(1.0,0.25)	N/A	0.5930	0.9640	1.0684	0.9875
	(1.0,0.50)	N/A	2.6262	0.8707	1.2475	0.9555
	(1.0,0.75)	N/A	6.9170	0.7438	1.4677	0.9067
Average performance***			0.9994	0.7726	1.6778	0.9763

[§] $N(\mu,\sigma)$ represents a normal distribution with mean μ and variance σ^2. *No exact solution is available for the test function. **If the exact solution is not available, the value from MCS using 10 million samples to approximate the exact value will be used. ***The average performance for each method is presented by averaging the predictions from each column.

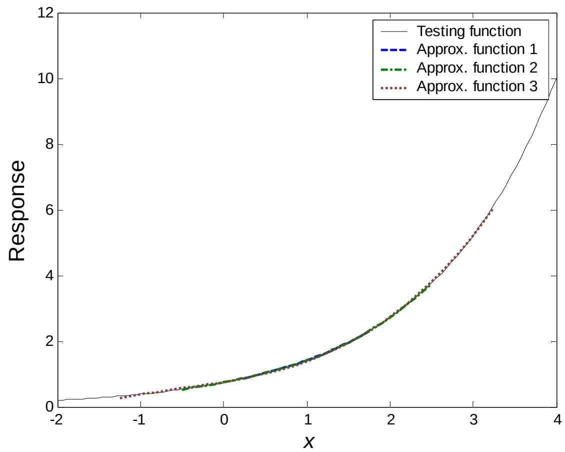
Table 4. The predictions of variance for different testing problems obtained from different techniques by assuming all random variables follow the given uniform distributions.

Given	$U(a,b)$[§]	Response variance by Exact approach	Ratio of the estimated value to the exact value MCS (1M)	First order approximation One pt.	Equally weighted 3 pt.	Optimally weighted 3 pt.
(1)	[0.0,0.50]	0.0326	0.9996	0.9973	1.0163	1.0000
	[0.0,1.00]	0.1889	0.9997	0.9926	1.0452	1.0000
	[0.0,2.00]	1.3556	1.0003	0.9836	1.1007	1.0000
	[1.0,2.00]	0.5222	1.0003	0.9973	1.0163	1.0000
(2)	[0.0,1.00]	0.1340	0.9994	0.8222	1.8882	0.9714
	[0.5,1.50]	0.8409	1.0001	0.9523	1.2051	0.9877
	[0.0,2.00]	3.8490	1.0001	0.8322	1.7237	0.9570
	[1.0,2.00]	2.4440	0.9996	0.9759	1.0958	0.9927
(3)	[0.0,0.50]	0.0124	1.0009	0.9794	1.1268	1.0000
	[0.0,1.00]	0.0283	0.9990	0.8546	1.8931	1.0000
	[0.0,2.00]	0.3358	0.9997	0.8041	2.2034	1.0000
	[1.0,2.00]	0.2383	1.0000	0.9827	1.1060	1.0000
(4)	[0.0,1.00]	0.3854	0.9998	1.0099	1.1713	1.0325
	[0.5,1.50]	0.0277	1.0008	0.2410	5.5881	0.9896
	[0.0,2.00]	0.3283	1.0001	0.0813	6.0813	0.9213
	[1.0,2.00]	0.2374	0.9986	0.9728	1.3845	1.0304
(5)	[0.0,1.00]	0.1523	1.0012	0.9723	1.0726	0.9863
	[0.0,2.00]	1.2682	1.0002	0.8945	1.2922	0.9502
	[1.0,2.00]	0.5587	0.9998	0.9723	1.0726	0.9863
	[1.0,3.00]	4.6532	0.9998	0.8945	1.2922	0.9502
(6)	[0.0,1.00]	0.3189	1.0002	0.8527	1.5551	0.9510
	[0.0,2.00]	9.2436	1.0001	0.8500	1.3036	0.9135
	[1.0,2.00]	6.3412	1.0005	0.9772	1.0268	0.9841
	[1.0,3.00]	62.0778	1.0012	0.9396	1.0386	0.9534
(7)	[0.0,0.50]	0.0122	0.9965	0.9554	1.2744	1.0001
	[0.0,1.00]	0.1538	0.9996	0.9434	1.3471	0.9999
	[0.0,2.00]	2.2917	1.0003	0.9391	1.3730	0.9998
	[1.0,3.00]	8.5687	0.9993	0.9836	1.0999	0.9999
(8)	[0.0,0.50]	0.0135	0.9995	0.9521	1.2415	0.9926
	[0.0,1.00]	0.1910	0.9998	0.9338	1.2693	0.9808
	[0.0,2.00]	3.4018	0.9981	0.8963	1.2288	0.9428
	[1.0,3.00]	17.8527	0.9992	0.9494	1.0364	0.9616
Average performance***			1.0000	0.8680	1.6791	0.9816

[§] $U(a,b)$ represents a uniform distribution with $x \in [a,b]$. ***The average performance for each method is presented by averaging the predictions from each column.



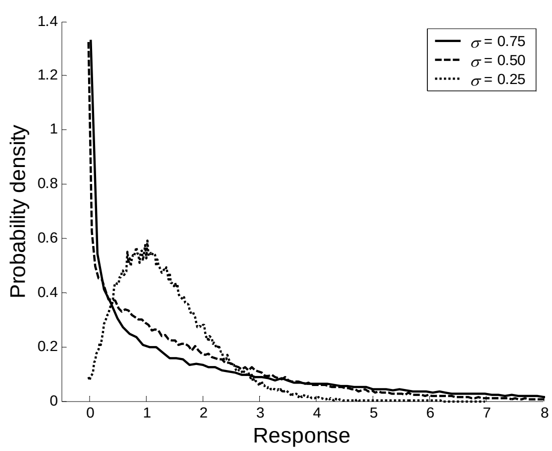
(a)



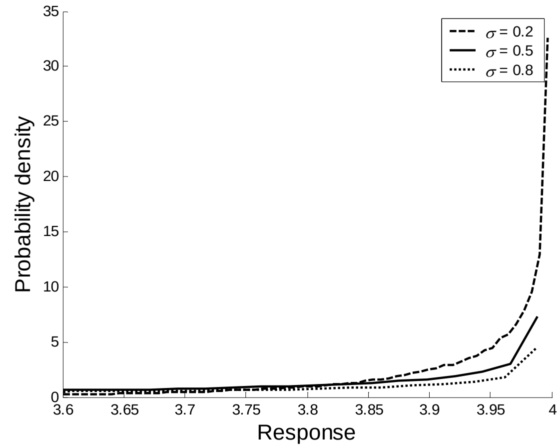
(b)

Figure 4. Show cubic polynomial approximations to specified regions of selected testing problems. (a) $g(x) = 4\sin(x + 0.5)$, and (b) $g(x) = 2\exp(0.65x - 0.3)$.

Considering the probability density function (PDF) of the response of selected functions obtained from MCS and shown in Fig. 5, we observe that the proposed approach works well even though the response distribution does not follow normal distributions. It is very clear that the PDF of Fig. 5(b) is far from being a normal distribution; nevertheless, the estimated variance still has a good agreement with the exact approach as evident from Table 3.



(a)



(b)

Figure 5. The probability density function of the testing functions using MCS at different levels of variance of the input variable *x*. (a) $y = 0.2x^3 + 1.5x^2 - 0.5x$, and (b) $g(x) = 4\sin(x + 0.5)$.

In addition, we test the variance estimation approach with a springback prediction model presented in Buranathiti and Cao (2004). In a straight flanging process shown in Fig. 6, springback, an elastic recovery due to uneven stress distributions, is the response of the model. In this work, we assume three parameters (Young's modulus *E*, sheet thickness *t*, and gap *g*) are associated with uncertainties. The problem is given by combining normal distributions and uniform distributions into single problem as shown in Table 5. Since there is no exact solution, an approximation from MCS is referred as the target value. We assume other deterministic parameters as follows: the die corner radius *R* is 5 mm, the initial yield stress is 375.5 MPa, the strength coefficient is 658.0 MPa, the hardening exponent is 0.14, and the Poisson's ratio is 0.3. As it can be seen in Table 5, the proposed approximation method is capable of handling problems with inputs following a combination of different distributions.

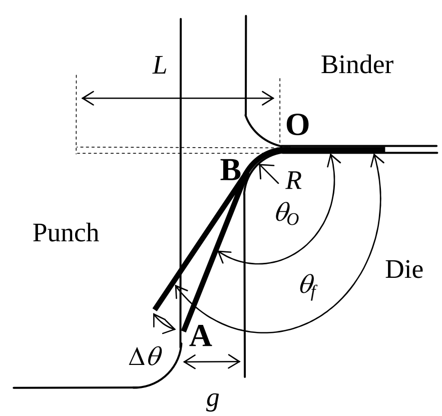


Figure 6. Schematic of a straight flanging process.

Table 5. The predictions of variance for springback prediction problems obtained from different techniques.

Given E (GPa)	t (mm.)	g (mm.)	Response variance by	Ratio of the estimated value to MCS		
				First order approximation		
$N(\mu,\sigma)$	$U(a,b)$	$U(a,b)$	MCS (1M)	One pt.	Equally weighted 3 pt.	Optimally weighted 3 pt.
[197.9,40.0]	[0.9,1.1]	[1.11,1.30]	1.5079	0.8335	5.0816	1.0261
[197.9,40.0]	[0.8,1.2]	[1.21,1.60]	2.3301	0.8586	4.6319	1.0092

$N(\mu,\sigma)$ represents a normal distribution with mean μ and variance σ^2. $U(a,b)$ represents a uniform distribution with $x \in [a,b]$.

The exact solution of the system for springback prediction, which cannot be written in a simple explicit function and is obviously more complicated than test functions used in Tables 3 and 4, is not available like most engineering problems. We can observe that the proposed method shown in the last column in Table 5 can well predict the variance in the springback prediction while the normal and uniform distributions are mixed together in the same system.

6. DISCUSSIONS AND CONCLUSIONS

The proposed approach is motivated by the fundamental belief that multiple observations are needed to be able to describe nonlinear systems and/or systems associated with large variations. Therefore, the traditional first order approximation needs an improvement in order to obtain better prediction accuracy while the computational effort is still relatively low. The weighted three-point-based strategy is proposed in this work. The necessary parameters for the weighted three-point-based strategy are directly derived and ready to be implemented for general systems when input variables follow normal distributions or uniform distributions. Only first order sensitivities of the system response of interest are needed to use this approach. The predicted results are significantly improved as we demonstrated in the example section.

For systems without explicit functions, the finite difference technique is a means to calculate the sensitivity of the system. However, concerning on the practical implementation, the size of the difference step needs to be properly small. However, in some cases, the finite difference technique may not be effective to estimate the derivative for noisy systems. In general, the approach is applicable for general systems.

The proposed three-point-based strategy requires ($6n$) function evaluations for the finite difference calculation and ($3n$) function evaluations for analytical derivatives for n-variable systems. In other words, the approach offers the computational complexity in a linear polynomial time scale O(n). Comparing to the traditional first order approximation, the significant improvement is achieved while the increase of the computational complexity is acceptable. With the additional effort, a significant improvement on the prediction accuracy is obtained. By considering the prediction robustness, we may not be satisfied with MCS using 50,000 samples for the given quadratic function shown in Fig. 1. Although the number of samples already is relatively high, the method still does not offer high prediction resolution due to the error of MCS. For the design using gradient search techniques, the proposed method offers better prediction resolution with a significantly less number of function evaluations over MCS.

The proposed approach is generally not only limited to the three-point-based approximation. The approach can be extended to multiple-point-based (4 points, 5 points, and etc.) approximations to improve the accuracy of the prediction if the trade-off between the increase of the accuracy and additional effort is met. The judgment of the number of required points is eventually up to the user and the available resources. However, we believe that the weighted three-point-based approach offers a good balance for general implementation. On the other note, the derivation of locations and weights for two widely used distributions (i.e., normal and uniform distributions) has been derived. For other distributions, e.g. Weibull, etc., the parameters may be obtained in a similar fashion. For some applications, different distributions may be needed for the approximation if a priori knowledge of the system is available. In addition, an improvement at the beginning of Taylor's expansion may be possible by using a more powerful approximation such as ones in Wang and Grandhi (1995).

We observe that the proposed weighted three-point-based approach can relatively better predict the variance even though the system response significantly deviates from normal distributions. Some comparisons are made by assuming that MCS in general can closely predict the exact solution when a large number of samples (e.g. 1 million) are implemented. On errors from the weighted three-point-based strategy, one reason may be that the cubic polynomial functions cannot well approximate the true system as we observe from Fig. 4. The smaller variation of systems usually leads to a better cubic polynomial approximation. In addition, a large variation causes a large error at the beginning of the approximation, i.e. Taylor expansion. Nevertheless, Tables 3-5 also show that the proposed method can well predict the variance at relatively large coefficient of variance ($\bar{x}/\sigma$ >25%). It is also understood that if the system of interest exhibits strongly nonlinear behavior and is associated with very large variation, Monte Carlo simulation (MCS) may be a proper technique for the uncertainty analysis. The systems with multiple variables and interaction between variables are also examined in Tables 3-5.

We clearly show in the column of 'Equally weight 3 pt.' that using arbitrary weight does not result a better estimation even if the additional effort is taken. If the additional effort is added, a proper procedure like the proposed approach needs to be taken. Considering the methods in the same efficient category, the proposed method better predicts the variance in every test problem shown in Tables 3-5. Particularly in the fourth problem in Tables 3-4, the first order approximation significantly underestimates the variance; therefore, the traditional first order approximation may potentially cause significant error on applications in robust design. In addition, it is known that the exact solution of the variance of multivariate problems is often very difficult to obtain. For generally difficult problems and/or black box models, the proposed approach clearly offers an effective and efficient prediction of variance.

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